CONTINUOUS RANDOM VARIABLES

Let $f(x)$ be given for $\alpha < x < \beta$.

$f(x)$ to be a pdf (probability distribution function) of a continuous random variable X;

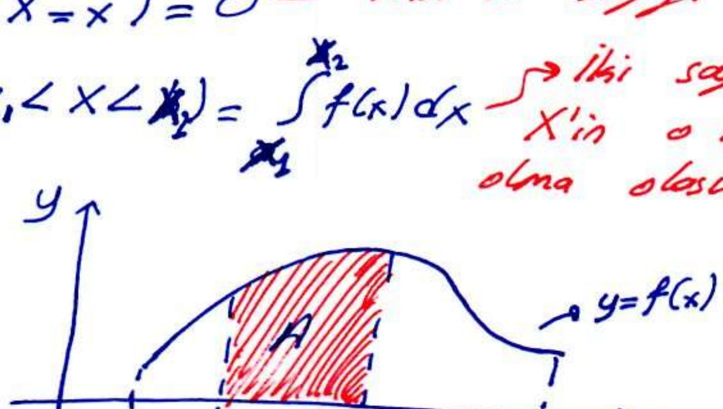
(i) $f(x) \geq 0 \rightarrow$ Nonnegative Probabilities

(ii) $\int_{\alpha}^{\beta} f(x)\,dx = 1 \rightarrow$ TOTAL AREA under the curve is 1.
(integral almayacağız, burada sadece ALAN'ın olasılığa eşit olduğunu bilmeniz yeterli.)

Also, for continuous case; We have

$P(X = x) = 0 \rightarrow$ X'in bir sayıya eşit olma olasılığı 0.

$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)\,dx \rightarrow$ iki sayı arasındaki alan, X'in o iki sayı arasında olma olasılığını verir.



$A = P(x_1 < X < x_2)$

$A = P(x_1 \leq X \leq x_2)$

↳ Continuous Case'de eşitliğin bir önemi yok.

Remember; $F(x) = P(X \leq x) \rightarrow$ olasılıkları buradan bulacağız, böylece integral almaya gerek yok.

$P(x_1 < X < x_2) = P(X < x_2) - P(X < x_1) = F(x_2) - F(x_1)$

$P(X > x_2) = 1 - P(X \leq x_2) = 1 - F(x_2)$

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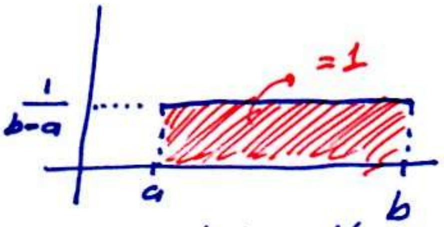
Uniform Distribution

$X \sim \text{Uniform}(a, b)$

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



* All equal length intervals are equally likely.

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

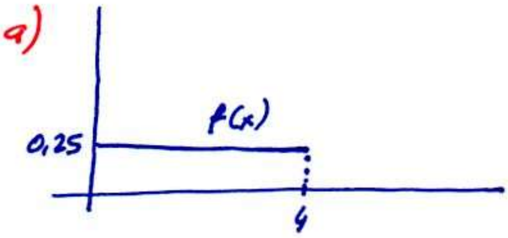
$$\mu = E(X) = \frac{b+a}{2} \qquad \sigma^2 = Var(X) = \frac{(b-a)^2}{12}$$

6.6 The jurisdiction of a rescue team includes emergencies occurring on a stretch of river that is 4 miles long. Experience has shown that the distance along this stretch, measured in miles from its northernmost point, at which an emergency occurs can be represented by a uniformly distributed random variable over the range 0 to 4 miles. Then, if X denotes the distance (in miles) of an emergency from the northernmost point of this stretch of river, its probability density function is

$$f(x) = \begin{cases} 0.25 & \text{for } 0 < x < 4 \\ 0 & \text{for all other } x \end{cases}$$

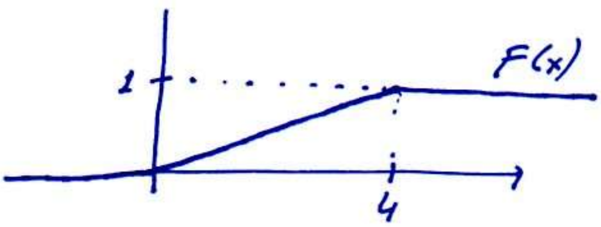
a. Draw the probability density function.
b. Find and draw the cumulative distribution function.
c. Find the probability that a given emergency arises within 1 mile of the northernmost point of this stretch of river.
d. The rescue team's base is at the midpoint of this stretch of river. Find the probability that a given emergency arises more than 1.5 miles from this base.

6.6) a)



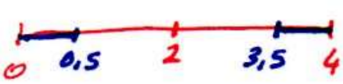
b)

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x}{4} & 0 \le x \le 4 \\ 1 & x > 4 \end{cases}$$



c) $P(X \le 1) = F(1) = \frac{1}{4} = \underline{0{,}25}$

d)



$P(X < 0{,}5) + P(X > 3{,}5) = F(0{,}5) + [1 - F(3{,}5)]$

$= \frac{0{,}5}{4} + 1 - \frac{3{,}5}{4} = \underline{0{,}25}$

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6.7 The incomes of all families in a particular suburb can be represented by a continuous random variable. It is known that the median income for all families in this suburb is $60,000 and that 40% of all families in the suburb have incomes above $72,000.

a. For a randomly chosen family, what is the probability that its income will be between $60,000 and $72,000?
b. Given no further information, what can be said about the probability that a randomly chosen family has an income below $65,000?

6.7) Median = 60000

$\Rightarrow P(X < 60000) = 0,5 = F(60000)$

Also; $P(X > 72000) = 0,4 = 1 - F(72000)$

$\Rightarrow F(72000) = 0,6$

a) $P(60000 < X < 72000)$

$= F(72000) - F(60000) = 0,6 - 0,5 = 0,1$

b) It is more than 0,5 because 65000 is more than median.

Remember; $W = a + bX$

$E(W) = E(a+bX) = a + b \cdot E(X)$

$Var(W) = Var(a+bX) = Var(bX) = b^2 \cdot Var(X)$

6.15 A charitable organization solicits donations by telephone. Employees are paid $60 plus 20% of the money their calls generate each week. The amount of money generated in a week can be viewed as a random variable with mean $700 and standard deviation $130. Find the mean and standard deviation of an employee's total pay in a week.

6.15) X: Money generated in a week

$E(X) = 700 \quad Var(X) = 130^2$

W: Money paid to employees

$W = 60 + 0,20 \cdot X$

Then; $E(W) = E(60 + 0,20X) = 60 + 0,20 \cdot \underbrace{E(X)}_{=700} = 200$

$Var(W) = Var(60 + 0,20X) = 0,20^2 \cdot Var(X) = 0,20^2 \cdot 130^2$

Std. dev. $(W) = \sqrt{0,20^2 \cdot 130^2} = 0,20 \cdot 130 = 26$

Standardized Random Variable;

$$\boxed{Z = \frac{X - \mu_X}{\sigma_X}}$$

Z shows, how many standard deviations is the unit X away from its mean μ_X.

We have; $\mu_Z = E(Z) = 0$; $\sigma_Z^2 = Var(Z) = 1$

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THE NORMAL DISTRIBUTION

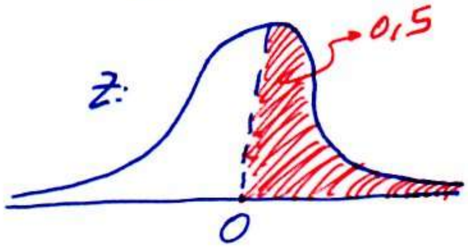
$$X \sim Normal(\mu ; \sigma^2)$$

$E(X) = \mu \qquad Var(X) = \sigma^2$

(we do not need $f(x)$ because we use Z-table to find probabilities about ~~Z~~)

The Standard Normal Distribution;

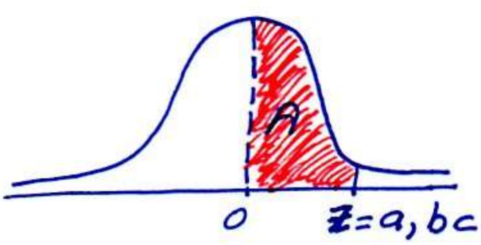
$$Z = \frac{X-\mu}{\sigma} : \quad Z \sim Normal(\mu = 0 ; \sigma^2 = 1^2)$$

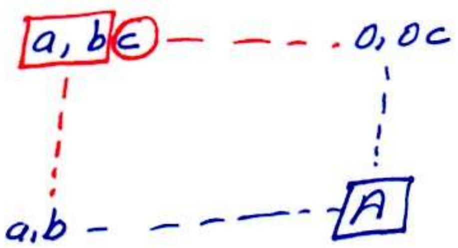


We have
(i) Total AREA = 1
(ii) Half AREA = 0,5
(iii) Symmetric AREA's are equal

$$P(-2,45 < Z < -1,23) = P(1,23 < Z < 2,45)$$

Finding Probabilities from Z-table.





$$A = P(0 < Z < a,bc)$$

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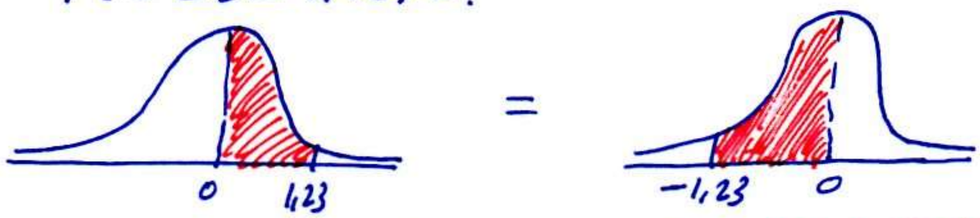
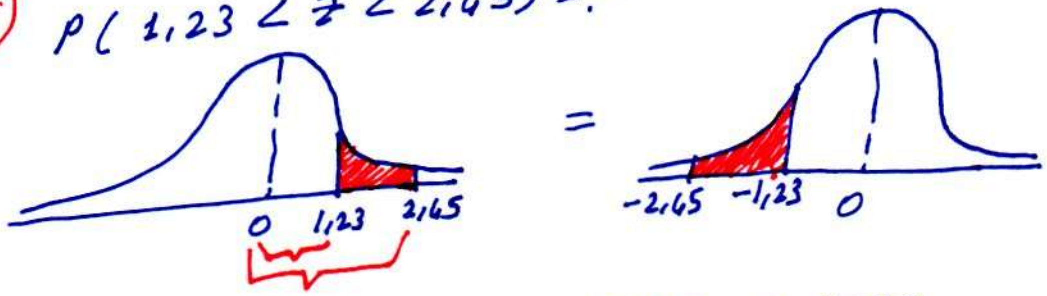
(I) $P(0 < z < 1{,}23) = ?$



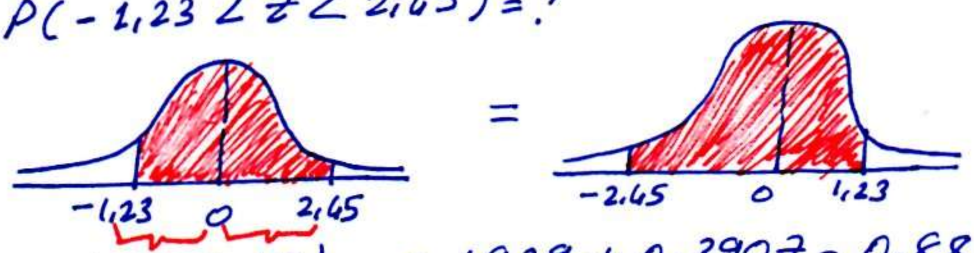
$P(0 < z < 1{,}23) = 0{,}3907 = P(-1{,}23 < z < 0)$

(II) $P(1{,}23 < z < 2{,}45) = ?$



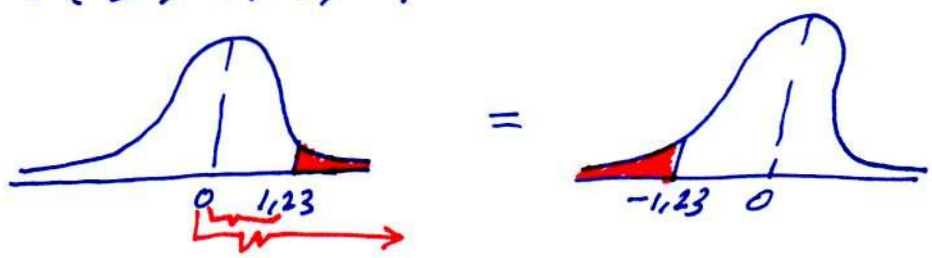
$P(1{,}23 < z < 2{,}45) = 0{,}4929 - 0{,}3907 = 0{,}1022$
$= P(-2{,}45 < z < -1{,}23)$

(III) $P(-1{,}23 < z < 2{,}45) = ?$



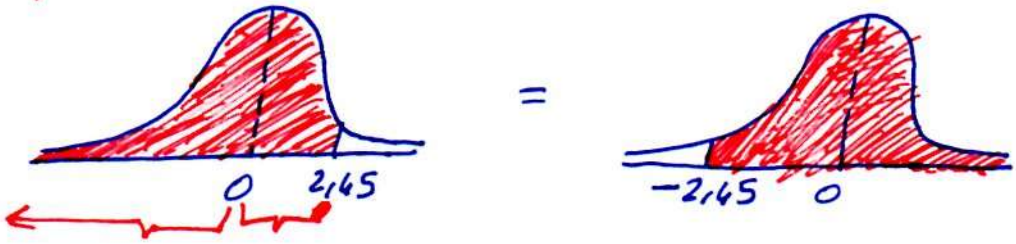
$P(-1{,}23 < z < 2{,}45) = 0{,}4929 + 0{,}3907 = 0{,}8836$
$= P(-2{,}45 < z < 1{,}23)$

(IV) $P(z > 1{,}23) = ?$



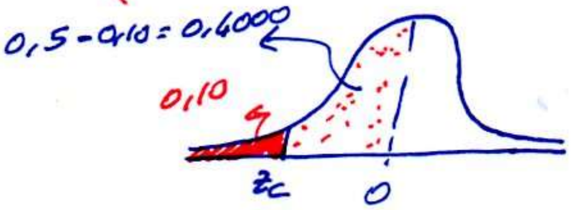
$P(z > 1{,}23) = 0{,}5 - 0{,}3907 = 0{,}1093$
$= P(z < -1{,}23)$

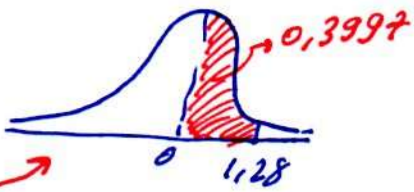
(V) $P(Z < 2,45) = ?$



$P(Z < 2,45) = 0,5 + 0,4929 = 0,9929$
$= P(Z > -2,45)$

(VI) (i) $P(Z < z_c) = 0,10 \Rightarrow z_c = ?$

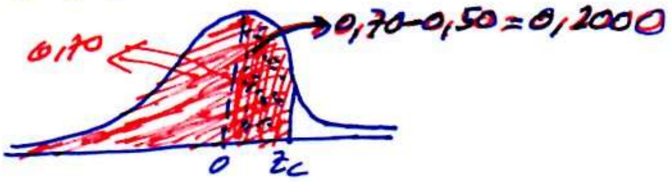




Nearest Number, $0,3997 \Rightarrow z_c = 1,28$

Negative Side $\Rightarrow \boxed{z_c = -1,28}$

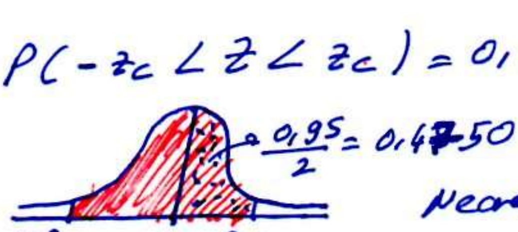
(ii) $P(Z < z_c) = 0,70 \Rightarrow z_c = ?$



Nearest Number, $0,1985 \Rightarrow z_c = 0,52$

Positive side $\Rightarrow \boxed{z_c = 0,52}$

(iii) $P(-z_c < Z < z_c) = 0,95 \Rightarrow z_c = ?$



Nearest Number $= 0,4750 \Rightarrow z_c = 1,96$

$\boxed{z_c = 1,96}$

* To find probabilities about $X \sim Normal(\mu, \sigma^2)$, we convert X to z by $z = \frac{X-\mu}{\sigma}$

* To find numbers for X satisfying a probability, we find z_c and put in the equation; $X_c = \mu + z_c \cdot \sigma$

6.31 Scores on an examination taken by a very large group of students are normally distributed with mean 700 and standard deviation 120.

a. An A is awarded for a score higher than 820. What proportion of all students obtain an A?

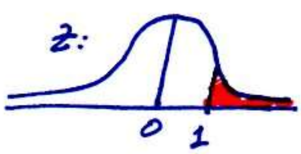
b. A B is awarded for scores between 730 and 820. An instructor has a section of 100 students who can be viewed as a random sample of all students in the large group. Find the expected number of students in this section who will obtain a B.

c. It is decided to give a failing grade to 5% of students with the lowest scores. What is the minimum score needed to avoid a failing grade?

6.31) $X \sim Normal(\mu = 700; \sigma^2 = 120^2)$

a) $P(X > 820) = P\left(\underbrace{\frac{X-\mu}{\sigma}}_{=z} > \frac{820-700}{120}\right)$

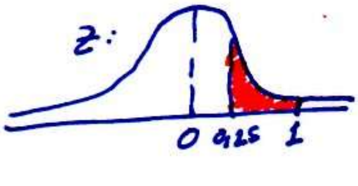
$= P(z > 1) = 0,5 - 0,3413 = 0,1587$



b) $n = 100$ students ; p = Prob. of achieving B

$p = P(730 < X < 820) = P\left(\frac{730-700}{120} < \underbrace{\frac{X-\mu}{\sigma}}_{=z} < \frac{820-700}{120}\right)$

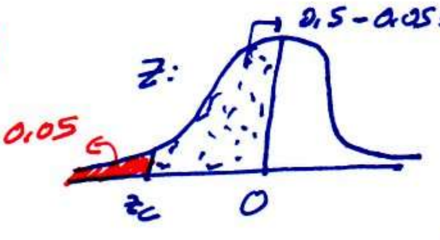
$= P(0,25 < z < 1) = 0,3413 - 0,0987 = 0,2626$



X: # of students achieving B

$E(X) = n \cdot p = 100 \cdot 0,2626 = \underline{24,26}$

c)



Nearest Areas: 0,4495 ; 0,4505

Corresponds to: 1,64 ; 1,65

Negative side; $z_c = -1,645$

$-1,645 = \frac{X_c - 700}{120} \Rightarrow X_c = 700 - 1,645 \cdot 120$

$\underline{X_c = 502,6}$

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6.33 A company can purchase raw material from either of two suppliers and is concerned about the amounts of impurity the material contains. A review of the records for each supplier indicates that the percentage impurity levels in consignments of the raw material follow normal distributions with the means and standard deviations given in the following table. The company is particularly anxious that the impurity level in a consignment not exceed 5% and wants to purchase from the supplier more likely to meet that specification. Which supplier should be chosen?

	Mean	Standard Deviation
Supplier A	4.4	0.4
Supplier B	4.2	0.6

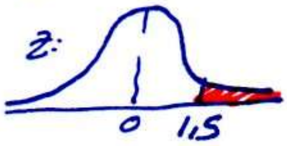
6.33) $X_A \sim Normal(\mu=4,4;\sigma^2=0,4^2)$

$X_B \sim Normal(\mu=4,2;\sigma^2=0,6^2)$

$P(X_A > 5) = P\left(\frac{X_A-\mu}{\sigma} > \frac{5-4,4}{0,4}\right)$

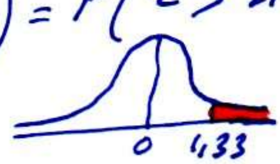
$= P(Z > 1,5) = 0,5 - 0,4332$

$= \underline{0,0668}$



$P(X_B > 5) = P\left(\frac{X_B-\mu}{\sigma} > \frac{5-4,2}{0,6}\right) = P(Z > 1,33) = 0,5 - 0,4082$

$= \underline{0,0918}$



First supplier should be choosen because it is less likely NOT to meet specification.

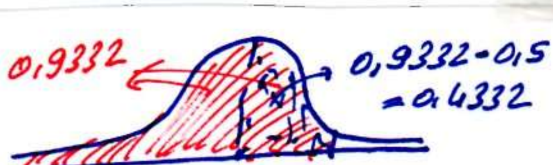
6.38 An economics test is taken by a large group of students. The test scores are normally distributed with mean 70, and the probability that a randomly chosen student receives a score less than 85 is 0.9332. Four students are chosen at random. What is the probability that at least one of them scores more than 80 points on this test?

6.38) $X \sim Normal(\mu=70;\sigma^2)$

$P(X < 85) = 0,9332$

$P\left(\frac{X-\mu}{\sigma} < \frac{85-70}{\sigma}\right) = 0,9332$

$P\left(Z < \underbrace{\frac{15}{\sigma}}_{z_c}\right) = 0,9332$



$P(0 < Z < z_c) = 0,4332$

$z_c = 1,5 \Rightarrow \frac{15}{\sigma} = 1,5 \Rightarrow \boxed{\sigma = 10}$

$p = P(X > 80) = P\left(\frac{X-\mu}{\sigma} > \frac{80-70}{10}\right) = P(Z > 1) = 0,5 - 0,3413$

$= 0,1587$

Y: # of students who get more than 80

$Y \sim Binomial(n=4;p=0,1587)$

$P(Y \geq 1) = 1 - P(Y=0) = \binom{4}{0} \cdot 0,1587^0 \cdot (1-0,1587)^4 = \underline{0,501}$

Normal Approximation to Binomial Distribution.

* For n is large and p is close to $0{,}5$, (preferably $np(1-p) > 9$), normal distribution provides a good approximation for binomial probabilities.

Set $\mu = np$; $\sigma^2 = np(1-p)$

for $X \sim \text{Binomial}(n; p)$

use $y \sim \text{Normal}(\mu; \sigma^2)$

Then; $P(X \geq b) = P(y > b - 0{,}5)$ and

$P(X \leq b) = P(y < b + 0{,}5)$.

6.47 A hospital finds that 25% of its bills are at least 1 month in arrears. A random sample of 450 bills was taken.

a. What is the probability that less than 100 bills in the sample were at least 1 month in arrears?

b. What is the probability that the number of bills in the sample at least 1 month in arrears was between 120 and 150 (inclusive)?

6.47) X: # of bills that are at least 1 month in arrears

$X \sim \text{Binomial}(n = 450; p = 0{,}25)$

$\mu = np = 450 \cdot 0{,}25 = 112{,}5$

$\sigma^2 = n \cdot p \cdot (1-p) = 450 \cdot 0{,}25 \cdot 0{,}75 = 84{,}4$ $> 9 \checkmark$

$\sigma = \sqrt{84{,}4} = 9{,}2$

$y \sim \text{Normal}(\mu = 112{,}5 ; \sigma^2 = 9{,}2^2)$

a) $P(X < 100) = P(X \leq 99) \cong P(y < 99{,}5) = P\left(\frac{y - \mu}{\sigma} < \frac{99{,}5 - 102{,}5}{9{,}2}\right)$

$= P(z < 0{,}33) = 0{,}5 + 0{,}1293 = 0{,}6293$

b) $P(120 \leq X \leq 150) = P(119{,}5 < y < 150{,}5) = P\left(\frac{119{,}5 - 102{,}5}{9{,}2} < z < \frac{150{,}5 - 102{,}5}{9{,}2}\right)$

$= P(1{,}85 < z < \underbrace{5{,}22}_{\text{ignore}}) = 0{,}5 - 0{,}4678 = 0{,}0332$

Exponential Distribution

$T \sim Exponential(\lambda)$ λ: Rate $=$ # of events / unit time

$$f(t) = \lambda . e^{-\lambda t}$$

$$F(t) = 1 - e^{-\lambda t}$$

$$E(T) = \frac{1}{\lambda} \; ; \quad Var(T) = \frac{1}{\lambda^2}$$

* X: # of events / unit time

T: Interarrival time of the events

$X \sim Poisson(\lambda) \iff T \sim Exponential(\lambda)$

6.55 A professor sees students during regular office hours. Times spent with students follow an exponential distribution with mean 10 minutes.

a. Find the probability that a given student spends less than 20 minutes with the professor.

b. Find the probability that a given student spends more than 5 minutes with the professor.

c. Find the probability that a given student spends between 10 and 15 minutes with the professor.

6.55) Mean $= \frac{1}{\lambda} = 10$ minutes

$\Rightarrow \lambda = 6$ students / hour.

$T \sim Exponential(\lambda = 6)$

a) $P(T < \frac{20}{60} \text{ hours}) = F(\frac{1}{3})$

$= 1 - e^{-6 \cdot \frac{1}{3}} = 0{,}8647$

b) $P(T > \frac{5}{60} \text{ hours}) = 1 - P(T < \frac{1}{12}) = 1 - F(12) = e^{-6 \cdot \frac{1}{12}}$

$= 0{,}6065$

c) $P(\frac{10}{60} < T < \frac{15}{60}) = P(T < \frac{15}{60}) - P(T < \frac{10}{60})$

$= F(\frac{1}{4}) - F(\frac{1}{6}) = (1 - e^{-6 \cdot \frac{1}{4}}) - (1 - e^{-6 \cdot \frac{1}{6}}) = e^{-1} - e^{-\frac{3}{2}}$

$= 0{,}1447$

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Jointly Distributed Continuous Random Variables

* Remember; $E(aX+bY+c) = a.E(X) + b.E(Y) + c$

$Var(aX+bY+c) = Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y)$

if X and Y are independent then $Cov(X,Y) = 0$

and $Var(aX+bY+c) = a^2 Var(X) + b^2 Var(Y)$

* If $X_1, X_2, \ldots, X_n$ are Normal random Variables, then $Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n$ is also a Normal random variable. Mean and Variance of Y is found as shown above.

6.69 It is estimated that in normal highway driving the number of miles that can be covered by automobiles of a particular model on 1 gallon of gasoline can be represented by a random variable with mean 28 and standard deviation 2.4. Sixteen of these cars, each with 1 gallon of gasoline, are driven independently under highway conditions. Find the mean and standard deviation of the average number of miles that will be achieved by these cars.

6.69) X_i: # of miles covered by using 1 gallon of gasoline

$E(X_i) = 28$; $Var(X_i) = 2{,}4^2$
$i = 1, 2, \ldots, 16$

Independence $\Rightarrow Cov(X_i, X_j) = 0$ for $i \neq j$

$$\bar{X} = \frac{\Sigma X_i}{n} \quad E(\bar{X}) = E\left(\frac{\Sigma X_i}{n}\right) = \frac{1}{n}.E(\Sigma X_i)$$

$$= \frac{1}{n}.E(X_1 + X_2 + \cdots + X_n)$$

$$= \frac{1}{n}.\underbrace{E(X_1)}_{=28} + \underbrace{E(X_2)}_{=28} + \cdots + \underbrace{E(X_n)}_{=28} = \frac{1}{\cancel{n}}.\cancel{n}.28 = 28$$

$$Var(\bar{X}) = Var\left(\frac{\Sigma X_i}{n}\right) = \frac{1}{n^2}.Var(\Sigma X_i) = \frac{1}{n^2}.Var(X_1) + \cdots + Var(X_n)$$

$$= \frac{1}{n^2}.n.2{,}4^2 = \frac{2{,}4^2}{n} \Rightarrow Std.dev(\bar{X}) = \frac{2{,}4}{\sqrt{n}} = \frac{2{,}4}{\sqrt{16}} = 0{,}6$$

6.70 Shirley Johnson, portfolio manager, has asked you to analyze a newly acquired portfolio to determine its mean value and variability. The portfolio consists of 50 shares of Xylophone Music and 40 shares of Yankee Workshop. Analysis of past history indicates that the share price of Xylophone Music has a mean of 25 and a variance of 121. A similar analysis indicates that Yankee has a mean share price of 40 with a variance of 225. Your best evidence indicates that the share prices have a correlation of +0.5.

a. Compute the mean and variance of the portfolio.
b. Suppose that the correlation between share prices was actually −0.5. Now what are the mean and variance of the portfolio?

6.70) X: Xylophone Music

$E(X) = 25;\ Var(X) = 121$

Y: Yankee Workshop

$E(Y) = 40;\ Var(Y) = 225$

W: Portfolio

$W = 50X + 40Y$

$\rho_{x,y} = 0{,}5$

Note that; $Cov(X,Y) = \rho_{x,y} \cdot \sigma_x \cdot \sigma_y = 0{,}5 \cdot \sqrt{121} \cdot \sqrt{225} = 82{,}5$

a) $E(W) = E(50X+40Y) = 50E(X) + 40E(Y) = 50 \cdot 25 + 40 \cdot 40$
$= 2850$

$Var(W) = Var(50X+40Y) = 50^2 \cdot Var(X) + 40^2 \cdot Var(Y) + 2 \cdot 50 \cdot 40 \cdot Cov(X,Y)$
$= 50^2 \cdot 121 + 40^2 \cdot 225 + 2 \cdot 50 \cdot 40 \cdot 82{,}5 = 992500$

$Std.Dev(W) = \sqrt{992500} = 996{,}24$

b) $E(W)$ will not change

$Var(W) = 50^2 \cdot 121 + 40^2 \cdot 225 - 2 \cdot 50 \cdot 40 \cdot 82{,}5 = 332500$

$Std.Dev.(W) = \sqrt{332500} = 576{,}63$

6.71 Prairie Flower Cereal has an annual sales revenue of $400,000,000. George Severn, a 58-year-old senior vice president, is responsible for production and sales of Nougy 93 Fruity cereal. Daily production in cases is normally distributed with a mean of 100 and a variance of 625. Daily sales in cases are also normally distributed with a mean of 100 and a standard deviation of 8. Sales and production have a correlation of 0.60. The selling price per case is $10. The variable production cost per case is $7. The fixed production costs per day are $250.

a. What is the probability that total revenue is greater than total costs on any day?
b. Construct a 95% acceptance interval for total sales revenue minus total costs.

6.71) X: Daily Production

$X \sim Normal(\mu = 100;\ \sigma^2 = 625)$

Y: Daily Sales

$Y \sim Normal(\mu = 100;\ \sigma^2 = 8^2)$

$\rho_{x,y} = 0{,}6$

Selling Price = \$10

Variable cost = \$7

Fixed cost = \$250

a) R: Total Revenue

$R = 10Y \;;\; E(R) = E(10Y) = 10E(Y) = 10 \cdot 100 = 1000$

$Var(R) = Var(10Y) = 10^2 \cdot Var(Y) = 10^2 \cdot 8^2 = 6400$

C: Total Cost.

$C = 250 + 7X \;;\; E(C) = E(250 + 7X) = 250 + 7 \cdot 100 = 950$

$Var(C) = Var(250 + 7X) = 7^2 \cdot Var(X) = 7^2 \cdot 625 = 30625$

Let $W = R - C$

$E(W) = E(R - C) = E(R) - E(C) = 1000 - 950 = 50$

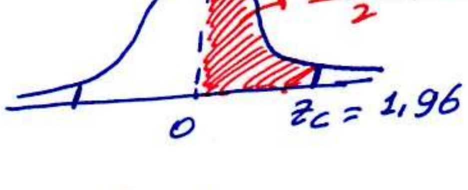
$Var(W) = Var(R - C) = Var(R) + Var(C) = 6400 + 30625 = 37025$

$\sigma_W = \sqrt{37025} = 192,42$

$P(R > C) = P(R - C > 0) = P(W > 0) = P\left(\frac{W - \mu_W}{\sigma_W} > \frac{0 - 50}{192,42}\right)$

$= P(Z > -0,26) = 0,5 + 0,1026 = 0,6026$

b)



$Z = \frac{W - \mu}{\sigma}$ $\qquad W_U = 50 + 1,96 \cdot 192,42 = 427,14$

$\qquad W_L = 50 - 1,96 \cdot 192,42 = -327,14$

$W = \mu + Z \cdot \sigma$

95% of the days, W will be in the interval $(-327,14 \;;\; 427,14)$

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