MATHEMATICAL STATISTICS Lecture Notes | CHAPTER 1

Random Variables & Probability Distributions

A random variable is a function defined on Sample Space which assigns each element of S to a real number.

Example: Let, a fair coin is tossed three times and let X be the number of Heads obtained. Then;

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$X(HHH) = 3$

$X(HHT) = X(HTH) = X(THH) = 2$

$X(HTT) = X(THT) = X(TTH) = 1$

$X(TTT) = 0$

The probability mass function (pmf) of a discrete random variable $f(x)$ assigns probabilities to the values X can take. We call a random variable X discrete if it takes discrete values (usually Integers) on a countable set. We have;

$$\underbrace{P(X}_{\text{Random Variable}} = \overbrace{x}^{\text{Numerical Value}}) = f(x)$$ and if $f(x)$ is a valid pmf,

(i) $f(x) \geq 0 \rightarrow$ Nonnegative Probabilities

(ii) $\sum_x f(x) = 1 \rightarrow$ Total Probability is 1.

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

1.48. For each of the following, find the constant c so that $f(x)$ satisfies the condition of being a p.d.f. of one random variable X.

(a) $f(x) = c(\frac{2}{3})^x$, $x = 1, 2, 3, \ldots$, zero elsewhere.

(b) $f(x) = cx$, $x = 1, 2, 3, 4, 5, 6$, zero elsewhere.

1.49. Let $f(x) = x/15$, $x = 1, 2, 3, 4, 5$, zero elsewhere, be the p.d.f. of X. Find $\Pr(X = 1 \text{ or } 2)$, $\Pr(\frac{1}{2} < X < \frac{5}{2})$, and $\Pr(1 \le X \le 2)$.

1.48) b) $f(x) = cx \quad x = 1, 2, 3, 4, 5, 6$

(i) $f(x) \geq 0$ for $c \geq 0$

(ii) $\sum_{x=1}^{6} f(x) = \sum_{x=1}^{6} cx = c.\sum_{x=1}^{6} x = c.\frac{6.7}{2} = 21c = 1$

$c = \frac{1}{21}$

a) $f(x) = c.\left(\frac{2}{3}\right)^x \quad x = 1, 2, 3, \ldots$

$\boxed{\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}}$

$f(x) \geq 0$ for $c \geq 0$

$\sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} c.\left(\frac{2}{3}\right)^x = c.\sum_{x=1}^{\infty}\left(\frac{2}{3}\right)^x = c.\left(\frac{1}{1-\frac{2}{3}}\right)$

$= 2c = 1$

$c = \frac{1}{2}$

1.49) $f(x) = \begin{cases} \frac{x}{15} & x = 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases}$

$P(X = 1 \text{ or } 2) = P(X = 1) + P(X = 2) = f(1) + f(2) = \frac{1}{15} + \frac{2}{15} = \frac{1}{5}$

$P(\frac{1}{2} < X < \frac{5}{2}) = P(0{,}5 < X < 2{,}5) = P(X = 1) + P(X = 2) = \frac{1}{5}$

$P(1 \leq X \leq 2) = P(X = 1) + P(X = 2) = \frac{1}{5}$

* Note that: $P(X \leq 3) = f(1) + f(2) + f(3)$

$P(X < 3) = f(1) + f(2)$

Cummulative Distribution Function (cdf)

The cummulative distribution function $F(x)$ is;

$$\boxed{F(x) = P(X \leq x)}$$

For a discrete random variable X, $\boxed{F(b) = \sum_{-\infty}^{b} f(x)}$

Given $F(x)$, it can be a cdf if it satisfies the following.

(i) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

(ii) $\lim_{h \to 0^+} F(x+h) = F(x) \Rightarrow F(x)$ is right continous

(iii) $a \leq b \Rightarrow F(a) \leq F(b) \Rightarrow F(x)$ is NONdecreasing

Note that for a and b are integers and X is a random variable that take integer values;

* $P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$

* $P(X = a) = f(a) = F(a) - F(a^-)$

1.50. Let $f(x)$ be the p.d.f. of a random variable X. Find the distribution function $F(x)$ of X and sketch its graph along with that of $f(x)$ if:
(a) $f(x) = 1$, $x = 0$, zero elsewhere.
(b) $f(x) = \frac{1}{3}$, $x = -1, 0, 1$, zero elsewhere.
(c) $f(x) = x/15$, $x = 1, 2, 3, 4, 5$, zero elsewhere.

1.51. Let us select five cards at random and without replacement from an ordinary deck of playing cards.
(a) Find the p.d.f. of X, the number of hearts in the five cards.
(b) Determine $\Pr(X \leq 1)$.

1.52. Let X equal the number of heads in four independent flips of a coin. Using certain assumptions, determine the p.d.f. of X and compute the probability that X is equal to an odd number.

1.50) a) $F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$

b)

x	-1	0	1
$f(x)$	$1/3$	$1/3$	$1/3$
$F(x)$	$1/3$	$2/3$	$3/3$

$$F(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{3} & -1 \leq x < 0 \\ \frac{2}{3} & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$

c) For $x = 1, 2, 3, 4, 5$;

$$F(b) = P(X \leq b) = \sum_{x=1}^{b} f(x) = \sum_{x=1}^{b} \frac{x}{15} = \frac{1}{15} \cdot \sum_{x=1}^{b} x = \frac{b \cdot (b+1)}{30}$$

Then;

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{\lfloor x \rfloor \cdot \lfloor x+1 \rfloor}{30} & 1 \leq x < 5 \\ 1 & x \geq 5 \end{cases}$$

1.51)

a) $\boxed{\begin{array}{c} 13 \text{ Hearts} \\ 39 \text{ NONhearts} \end{array}}$ Select $n=5$ cards

52 Cards

X: Number of hearts

$$f(x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}$$

for $x = 0, 1, 2, 3, 4, 5$ (=0 otherwise)

b) $P(X \leq 1) = f(0) + f(1) = \dfrac{\binom{13}{0}\binom{39}{5}}{\binom{52}{5}} + \dfrac{\binom{13}{1}\binom{39}{4}}{\binom{52}{5}} = 0,633$

1.52) If X is a fair coin, $f(x) = \binom{4}{x} \cdot \left(\frac{1}{2}\right)^{x} \cdot \left(\frac{1}{2}\right)^{4-x} = \binom{4}{x} \cdot \left(\frac{1}{2}\right)^{4}$

$P(X \text{ is odd}) = f(1) + f(3) = \binom{4}{1} \cdot \left(\frac{1}{2}\right)^{4} + \binom{4}{3} \cdot \left(\frac{1}{2}\right)^{4} = 0,5$

1.53. Let X have the p.d.f. $f(x) = x/5050$, $x = 1, 2, 3, \ldots, 100$, zero elsewhere.

(a) Compute $\Pr(X \le 50)$.

(b) Show that the distribution function of X is $F(x) = [x]([x] + 1)/10100$, for $1 \le x \le 100$, where $[x]$ is the greatest integer in x.

1.54. Let a bowl contain 10 chips of the same size and shape. One and only one of these chips is red. Continue to draw chips from the bowl, one at a time and at random and without replacement, until the red chip is drawn.

(a) Find the p.d.f. of X, the number of trials needed to draw the red chip.

(b) Compute $\Pr(X \le 4)$.

1.55. Cast a die a number of independent times until a six appears on the up side of the die.

(a) Find the p.d.f. $f(x)$ of X, the number of casts needed to obtain that first six.

(b) Show that $\sum_{x=1}^{\infty} f(x) = 1$.

(c) Determine $\Pr(X = 1, 3, 5, 7, \ldots)$.

(d) Find the distribution function $F(x) = \Pr(X \le x)$.

1.53) a) $P(X \le 50) = \sum_{x=1}^{50} f(x) = \sum_{x=1}^{50} \frac{x}{5050} = \frac{1}{5050} \cdot \sum_{x=1}^{50} x = \frac{1}{5050} \cdot \frac{50 \cdot 51}{2} = 0{,}252$

b) For $x = 1, 2, \ldots, 100$;

$F(x) = P(X \le x) = \sum_{w=1}^{x} f(w) = \sum_{w=1}^{x} \frac{x}{5050} = \frac{1}{5050} \cdot \sum_{w=1}^{x} f(w) = \frac{1}{5050} \cdot \frac{\cancel{x} \cdot (x+1)}{2}$

Then; $F(x) = \begin{cases} 0 & x < 1 \\ \frac{[x] \cdot [x+1]}{10100} & 0 \le x \le 100 \\ 1 & x > 100 \end{cases}$

1.54) a) $f(x) = P(X = x) = P(\text{First } x-1 \text{ trials } \text{Nonred}, x^{th} \text{ is red})$

$= \frac{9}{10} \cdot \frac{8}{9} \cdot \ldots \cdot \frac{9-x+2}{10-x+1} \cdot \frac{1}{10-x} = \frac{P(9, x-1)}{P(10, x)} = \frac{1}{10}$ for $x = 1, 2, \ldots, 10$

Permutation

$P(X \le 4) = \sum_{x=1}^{4} f(4) = \frac{1}{10} + \frac{9}{10} \cdot \frac{1}{9} + \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} + \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{1}{6} = 4 \cdot \frac{1}{10} = 0{,}4$

A random variable is continous if it takes values in a real interval. The probability density function (pdf) of a continous random variable $f(x)$ is used to find probability of X to be between two numbers. $f(x)$ is called a density function because it does NOT assign probabilities to X. Namely,

$$P(X = x) = 0$$

$$P(a < X < b) = \int_a^b f(x)dx$$

The area under the curve $f(x)$ between a and b is the probability of X between a and b. $f(x)$ to be a valid pdf, it should satisfy

(i) $f(x) \geq 0 \rightarrow$ Nonnegative Probabilities

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1 \rightarrow$ Total Probability is 1.

Example: Find c if $f(x) = \begin{cases} cx^3 & 0 < x < 1 \\ 0 & o.w \end{cases}$ is a pdf.

Answer: (i) $f(x) \geq 0$ for $c \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} cx^3 dx = \int_0^1 cx^3 dx = c \cdot \int_0^1 x^3 dx$

$$= c \cdot \frac{x^4}{4}\Big]_0^1 = c \cdot \left(\frac{1}{4} - \frac{0}{4}\right) = \frac{c}{4} = 1$$

$$c = 4$$

$$f(x) = \begin{cases} 4x^3 & 0 < x < 1 \\ 0 & o.w \end{cases}$$

1.58. Let the probability set function $P(A)$ of the random variable X be $P(A) = \int_A f(x)\,dx$, where $f(x) = 2x/9$, $x \in \mathscr{A} = \{x : 0 < x < 3\}$. Let $A_1 = \{x : 0 < x < 1\}$, $A_2 = \{x : 2 < x < 3\}$. Compute $P(A_1) = \Pr[X \in A_1]$, $P(A_2) = \Pr(X \in A_2)$, and $P(A_1 \cup A_2) = \Pr(X \in A_1 \cup A_2)$.

1.58)
$$f(x) = \begin{cases} \frac{2x}{9} & 0 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

$$P(A_1) = P(0 < X < 1) = \int_0^1 f(x)dx = \int_0^1 \frac{2x}{9}dx = \frac{2}{9}\int_0^1 x\,dx = \frac{2}{9}$$

$$P(A_2) = P(2 < X < 3) = \int_2^3 \frac{2x}{9}dx = \frac{2}{9}\,\frac{x^2}{2}\Big]_2^3 = \frac{1}{9}(3^2 - 2^2) = \frac{5}{9}$$

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) = \frac{2}{9} + \frac{5}{9} = \frac{7}{9}$$

1.64. For each of the following probability density functions of X, compute $\Pr(|X| < 1)$ and $\Pr(X^2 < 9)$.
(a) $f(x) = x^2/18$, $-3 < x < 3$, zero elsewhere.
(b) $f(x) = (x + 2)/18$, $-2 < x < 4$, zero elsewhere.

1.64) a)
$$f(x) = \begin{cases} \frac{x^2}{18} & -3 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$

$$P(|X| < 1) = P(-1 < X < 1) = \int_{-1}^{1} \frac{x^2}{18}dx = \frac{1}{18}\,\frac{x^3}{3}\Big]_{-1}^{1} = \frac{1}{27}$$

$$P(X^2 < 9) = P(-3 < X < 3) = \int_{-3}^{3} f(x)dx = 1$$

b)
$$f(x) = \begin{cases} \frac{x+2}{18} & -2 < x < 4 \\ 0 & \text{o.w} \end{cases}$$

$$P(|X| < 1) = \int_{-1}^{1} \frac{x+2}{18}dx = \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-1}^{1} = \frac{1}{18}\left(\frac{1}{2} + 2 - \frac{1}{2} + 2\right) = \frac{2}{9}$$

$$P(X^2 < 9) = \int_{-3}^{3} f(x)dx = \int_{-2}^{3} f(x)dx = \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-2}^{3} = \frac{1}{18}\left[\left(\frac{3^2}{2} + 6\right) - \left(\frac{2^2}{2} - 4\right)\right] = \frac{7}{12}$$

7

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

Cummulative Distribution Function

we replace Σ by $\int$ to find cdf of a continuous random variable X.

$$\boxed{F(x) = \int_{-\infty}^{w} f(w)\,dw}$$

1.69. Find the distribution function $F(x)$ associated with each of the following probability density functions. Sketch the graphs of $f(x)$ and $F(x)$.
(a) $f(x) = 3(1-x)^2$, $0 < x < 1$, zero elsewhere.
(b) $f(x) = 1/x^2$, $1 < x < \infty$, zero elsewhere.
(c) $f(x) = \frac{1}{3}$, $0 < x < 1$ or $2 < x < 4$, zero elsewhere.

1.69) a) $f(x) = \begin{cases} 3.(1-x)^2 & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$

For $0<x<1$; $F(x) = \int_0^{x} f(w)\,dw = \int_0^{x} 3.(1-w)^2\,dw = -3.\int_1^{1-x} t^2\,dt$

$1-w=t$
$-dw=-dt$

$$= 3\int_{1-x}^{1} t^2\,dt = 3.\frac{t^3}{3}\Big]_{1-x}^{1} = 1-(1-x)^3$$

$$F(x) = \begin{cases} 0 & x \leq 0 \\ 1-(1-x)^3 & 0<x<1 \\ 1 & x \geq 1 \end{cases}$$

b) For $x>1$, $F(x) = \int_1^{x} f(w)\,dw = \int_1^{x} \frac{1}{w^2}\,dw = \cancel{\tfrac{1}{3}}.\frac{-1}{w^3}\Big]_1^{x} = -\frac{1}{x^3}+1$

$$F(x) = \begin{cases} 0 & x<1 \\ 1-\frac{1}{x^3} & x \geq 1 \end{cases}$$

c) $f(x) = \begin{cases} \frac{1}{3} & 0 < x < 1 \text{ OR } 2 < x < 4 \end{cases}$

For $0 < x \leq 1$; $F(x) = \int_0^x \frac{1}{3}\,dw = \frac{x}{3}$ and $F(1) = \frac{1}{3}$

For $2 < x < 4$; $F(x) = \frac{1}{3} + \int_2^x \frac{1}{3}\,dw = \frac{1}{3} + \frac{1}{3} w\Big]_2^x = \frac{1}{3} + \left(\frac{x}{3} - \frac{2}{3}\right) = \frac{x}{3} - \frac{1}{3}$

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{3} & 0 \leq x < 1 \\ \frac{1}{3} & 1 \leq x < 2 \\ \frac{x}{3} - \frac{1}{3} & 2 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

1.71. Given the distribution function

$$\begin{aligned} F(x) &= 0, & x &< -1, \\ &= \frac{x+2}{4}, & -1 &\leq x < 1, \\ &= 1, & 1 &\leq x. \end{aligned}$$

Sketch the graph of $F(x)$ and then compute: (a) $\Pr(-\frac{1}{2} < X \leq \frac{1}{2})$; (b) $\Pr(X=0)$; (c) $\Pr(X=1)$; (d) $\Pr(2 < X \leq 3)$.

1.71) a) $P(-\frac{1}{2} < x \leq \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{\frac{1}{2}+2}{4} - \frac{-\frac{1}{2}+2}{4} = 0{,}625$

b) $P(X=0) = 0$ c) $P(X=1) = 0$ (Since X is continuous)

d) $P(2 < X \leq 3) = F(3) - F(2) = 1 - 1 = 0$

Example: a) Show that $F(x)=\begin{cases} 0 & x<1 \\ 1-x^{-k} & x\geq 1 \end{cases}$ is a valid cdf for $k>0$

Answer: (i) $\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} 0 = 0$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} (1-x^{-k}) = 1 - \underbrace{\lim_{x\to\infty} \frac{1}{x^k}}_{=0} = 1$$

(ii) $\lim_{h\to 0^+} F(x+h) = \lim_{h\to 0^+} 1-(x+h)^{-k} = 1-x^{-k} = F(x)$

($F(x)$ is right continuous for $x=1$ also)

(iii) $F(a) \leq F(b) \Rightarrow 1-a^{-k} \leq 1-b^{-k}$

$$b^{-k} \leq a^{-k}$$

$$b^k \geq a^k \Rightarrow a \leq b$$

$F(x)$ is NON decreasing.

Then, $F(x)$ is a valid cdf.

b) Find $P(X<7)$ and $P(X>10)$

Answer: $P(X<7) = P(X\leq 7) = F(7) = 1-7^{-k}$

$P(X>10) = 1-P(X\leq 10) = 1-F(10) = 10^{-k}$

c) find $f(x)$

Answer: $f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}(1-x^{-k}) = k.x^{-(k+1)}$

pdf of a random variable $Y=g(X)$

given pdf of a random variable $X: f_X(x)$, we can find pdf of $Y=g(X): f_Y(x)$ using the following technique, which is called distribution-function technique.

(i) Find $F_X(x) = \int_{-\infty}^{x} f_X(w)\,dw$

(ii) Find $F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$

$= P(X \leq g^{-1}(y)) = F_X(g^{-1}(y))$

(iii) Find $f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$

1.73. Let $f(x)=x/6$, $x=1, 2, 3$, zero elsewhere, be the p.d.f. of X. Find the distribution function and the p.d.f. of $Y=X^2$.

Hint: Note that X is a random variable of the discrete type.

1.74. Let $f(x)=(4-x)/16$, $-2<x<2$, zero elsewhere, be the p.d.f. of X.
(a) Sketch the distribution function and the p.d.f. of X on the same set of axes.
(b) If $Y=|X|$, compute $\Pr(Y \leq 1)$.
(c) If $Z=X^2$, compute $\Pr(Z \leq \frac{1}{4})$.

1.75. Let X have the p.d.f. $f(x)=2x$, $0<x<1$, zero elsewhere. Find the distribution function and p.d.f. of $Y=X^2$.

1.76. Let X have the p.d.f. $f(x)=4x^3$, $0<x<1$, zero elsewhere. Find the distribution function and p.d.f. of $Y=-2\ln X^4$.

1.73)

X	1	2	3
$f(x)$	$1/6$	$2/6$	$3/6$

$Y=X^2$	1	4	9
$f_Y(y)$	$1/6$	$2/6$	$3/6$

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1.74) a) $f(x)=\begin{cases}\frac{4-x}{16} & -2<x<2\\ 0 & \text{o.w}\end{cases}$

For $-2<x<2$:

$$F(x)=\int_{-2}^{x}\frac{4-w}{16}dw=-\frac{1}{16}\int_{6}^{4-x}t\,dt=\frac{1}{16}\cdot\frac{t^2}{2}\Big]_{4-x}^{6}=\frac{36-(4-x)^2}{32}$$

$4-w=t$
$-dw=dt$

$$F(x)=\begin{cases}0 & x\le -2\\ \frac{36-(4-x)^2}{32} & -2<x<2\\ 1 & x\ge 2\end{cases}$$

b) $P(Y\le 1)=P(|X|\le 1)=P(-1\le X\le 1)=F(1)-F(-1)$

$$=\frac{36-9}{32}-\frac{36-25}{32}=\frac{1}{2}$$

$$P(Z\le \tfrac{1}{4})=P(X^2\le\tfrac{1}{4})=P(-\tfrac{1}{2}<X<\tfrac{1}{2})=F(\tfrac{1}{2})-F(-\tfrac{1}{2})=\frac{8}{32}=\frac{1}{4}$$

1.75) $f(x)=\begin{cases}2x & 0<x<1\\ 0 & \text{o.w.}\end{cases}\Rightarrow F(x)=\begin{cases}0 & x\le 0\\ \frac{x^2}{2} & 0<x<1\\ 1 & x\ge 1\end{cases}$

$$F_Y(y)=P(Y\le y)=P(X^2\le y)=P(-\sqrt{y}\le X^2\le\sqrt{y})$$
$$=F(\sqrt{y})-F(-\sqrt{y})=F(\sqrt{y})=\frac{y}{2}\quad : 0<y<1$$

$$f_Y(y)=\frac{d}{dy}\left(\frac{y}{2}\right)=\frac{1}{2}\qquad f_Y(y)=\begin{cases}\frac{1}{2} & 0<y<1\\ 0 & \text{o.w.}\end{cases}$$

1.76)

$$f(x)=\begin{cases}4x^3 & 0<x<1\\ 0 & o.w\end{cases} \Rightarrow F(x)=\begin{cases}0 & x\le 0\\ x^4 & 0<x<1\\ 1 & x\ge 1\end{cases}$$

$y=-2\ln X^4$

$$F_y(y)=P(Y\le y)=P(-2\ln X^4\le y)=P(\ln X^4\ge -0{,}5y)$$

$$=P(X^4\ge e^{-0{,}5y})=P(X\ge e^{-0{,}125y})=1-P(X\le e^{-0{,}125y})$$

$$=1-F(e^{-0{,}125y})=1-(e^{-0{,}125y})^4=1-e^{-y/2} \quad y\ge 0$$

$(x=0^+\Rightarrow y\to+\infty$ and $x=1\Rightarrow y=0)$

$$f_y(y)=\frac{d}{dy}F_y(y)=\frac{d}{dy}\left[1-e^{-y/2}\right]=\frac{1}{2}\cdot e^{-y/2}$$

Expectation of a Random Variable

Expected Value, Mean and Average have the same meaning. Expected Value of a Random Variable is its long run average.

Let, $\sum_x |x|.f(x)$ converges to a limit for discrete case and $\int_{-\infty}^{\infty}|x|.f(x)$ converges to a limit for continuous case. Expectation of a random variable is;

$$E(X)=\begin{cases}\sum_x x.f(x) & \text{if } X \text{ is discrete}\\ \int_{-\infty}^{\infty} x.f(x)dx & \text{if } X \text{ is continuous.}\end{cases}$$

Likewise, expected value of a function of random variable $X : g(x)$ is found by;

$$E[g(x)] = \begin{cases} \sum_x g(x) \cdot f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) \cdot f(x) & \text{if } X \text{ is continuous.} \end{cases}$$

1.80. Let X have the p.d.f. $f(x) = (x + 2)/18$, $-2 < x < 4$, zero elsewhere. Find $E(X)$, $E[(X + 2)^3]$, and $E[6X - 2(X + 2)^3]$.

1.80)

$$f(x) = \begin{cases} \frac{x+2}{18} & -2 < x < 4 \\ 0 & \text{o.w.} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{-2}^{4} x \cdot \frac{x+2}{18} dx = \frac{1}{18} \int_{-2}^{4} (x^2 + 2x) = \frac{1}{18} \left(\frac{x^3}{3} - x^2 \right)_{-2}^{4}$$

$$= \frac{1}{18} \left[\left(\frac{4^3}{3} - 4^2 \right) - \left(-\frac{2^3}{3} - 2^2 \right) \right] = \frac{12}{18} = \frac{2}{3}$$

$$E[(X+2)^3] = \int_{-2}^{4} (x+2)^3 \cdot \frac{x+2}{18} dx = \frac{1}{18} \int_{-2}^{4} (x+2)^4 dx = \frac{1}{18} \int_{0}^{6} t^4 dt$$

$x+2 = t$
$dx = dt$

$$= \frac{1}{18} \left(\frac{t^5}{5} \right)_0^6 = 86{,}4$$

$$E[6X - 2(X+2)^3] = 6E(X) - 2 \cdot E[(X+2)^3]$$

$$= 6 \cdot \frac{2}{3} - 2 \cdot 86{,}4 = -168{,}8$$

The last calculation is done by Linearity property of Expectation operator. This follows because both Σ and $\int$ are linear operators. For a and b are constant, we have

$$\boxed{E[a.g_1(X) + b.g_2(X)] = a.E[g_2(X)] + b.E[g_2(X)]}$$

Note that this is NOT true for Multiplication.

$$E[g_1(X).g_2(X)] \neq E[g_1(X)].E[g_2(X)]$$

1.84. Let X have the p.d.f. $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere. Consider a random rectangle whose sides are X and $(1 - X)$. Determine the expected value of the area of the rectangle.

1.85. A bowl contains 10 chips, of which 8 are marked \$2 each and 2 are marked \$5 each. Let a person choose, at random and without replacement, 3 chips from this bowl. If the person is to receive the sum of the resulting amounts, find his expectation.

1.84) $A = X.(1-X)$

$$E(A) = E[X.(1-X)] = E(X - X^2) = E(X) - E(X^2)$$

$$E(X) = \int_0^1 x 3x^2 dx = 3.\int_0^1 x^3 dx = 3.\frac{x^4}{4}\Big]_0^1 = \frac{3}{4}$$

$$E(X^2) = \int_0^1 x^2.3x^2 dx = 3\int_0^1 x^4 dx = 3.\frac{x^5}{5}\Big]_0^1 = \frac{3}{5}$$

Then, $E(A) = \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$

1.85) $\boxed{\begin{array}{c}\$2 \to 8 \\ \$5 \to 2\end{array}}$ $\to$ Select $n=3$ chips

10 chips

Let $T = X_1 + X_2 + X_3$

$$P(T=24) = \frac{\binom{8}{3}\binom{2}{0}}{\binom{10}{3}} = 0{,}467$$

$$P(T=18) = \frac{\binom{8}{2}\binom{2}{1}}{\binom{10}{3}} = 0{,}467 \quad \text{and} \quad P(T=12) = \frac{\binom{8}{1}\binom{2}{2}}{\binom{10}{3}} = 0{,}066$$

T	24	18	12
$f_T(t)$	0,467	0,467	0,066

Then, $E(T) = \sum t \cdot f_T(t) = 24 \cdot 0{,}467 + 18 \cdot 0{,}467 + 12 \cdot 0{,}066$

$$= \underline{20{,}406}$$

1.87. Let $f(x) = 2x$, $0 < x < 1$, zero elsewhere, be the p.d.f. of X.
(a) Compute $E(1/X)$.
(b) Find the distribution function and the p.d.f. of $Y = 1/X$.
(c) Compute $E(Y)$ and compare this result with the answer obtained in part (a).
Hint: Here $\mathscr{A} = \{x : 0 < x < 1\}$, find $\mathscr{B}$.

1.88. Two distinct integers are chosen at random and without replacement from the first six positive integers. Compute the expected value of the absolute value of the difference of these two numbers.

1.87) a) $E\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x} \cdot 2x\,dx = \int_0^1 2\,dx = 2x\Big]_0^1 = 2$

b) $F(x) = x^2 \quad 0 < x < 1$

$$F_Y(y) = P(Y \le y) = P\left(\frac{1}{X} \le y\right) = P\left(X \ge \frac{1}{y}\right) = 1 - P\left(X \le \frac{1}{y}\right)$$

$$= 1 - F\left(\frac{1}{y}\right) = 1 - \frac{1}{y^2} \quad 1 < y < \infty$$

$$f_Y(y) = \frac{d}{dy}\left[1 - \frac{1}{y^2}\right] = \frac{2}{y^3} \quad y > 1$$

c) $$E(Y) = \int_1^{\infty} y \cdot \frac{2}{y^3}\, dy = 2\int_1^{\infty} \frac{1}{y^2}\, dy = 2\left[-\frac{1}{y}\right]_1^{\infty} = 2(0-(-1)) = 2$$

1.88) Table for X: Diffeence of two integers

II \ I	1	2	3	4	5	6
1	—	1	2	3	4	5
2	1	-	1	2	3	4
3	2	1	-	1	2	3
4	3	2	1	-	1	2
5	4	3	2	1	-	1
6	5	4	3	2	1	-

X	1	2	3	4	5
f(x)	$\frac{10}{30}$	$\frac{8}{30}$	$\frac{6}{30}$	$\frac{4}{30}$	$\frac{2}{30}$

(or; $f(x) = \begin{cases} \frac{6-x}{15} & x = 1,2,3,4,5 \\ 0 & \text{o.w.} \end{cases}$)

$$\sum_1^n x = \frac{n.(n+1)}{2}$$

$$\sum_1^n x^2 = \frac{n(n+1)(2n+1)}{6}$$

$$E(X) = \sum_x x f(x) = \sum_{x=1}^{5} x \cdot \frac{6-x}{15} = \frac{1}{15}\left(6\sum_{x=1}^{5} x - \sum_{x=1}^{5} x^2\right)$$

$$= \frac{1}{15}\left(6 \cdot \frac{5.\cancel{6}^{3}}{\cancel{2}} - \frac{5.\cancel{6}.11}{\cancel{6}}\right) = \frac{35}{15} = 2.33$$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

Example: Let $f(x)=\dfrac{1}{\pi(1+x^2)}$ $\quad -\infty<x<\infty$

Find $E(X)$ if it exists.

Answer: Remember, $E(X)$ exists only if $\int_{-\infty}^{\infty}|x|f(x)dx$ converges to a finite limit. We have,

$$\int_{-\infty}^{\infty}|x|f(x)dx=\int_{-\infty}^{\infty}|x|.\frac{1}{\pi(1+x^2)}dx=2.\int_{0}^{\infty}\frac{|x|}{\pi(1+x^2)}dx$$ (since we want to integrate an even function)

$$=\frac{1}{\pi}\int_{0}^{\infty}\frac{2x}{1+x^2}dx=\frac{1}{\pi}.\lim_{k\to\infty}\ln(1+x^2)\Big]_0^k=\frac{1}{\pi}.\lim_{k\to\infty}\ln(k)\to\infty$$

So, $E(X)$ does NOT exist.

Some Special Expectations:

$\mu=E(X)$: Mean of a random Variable X.

$\sigma^2=Var(X)$: Variance of a random Variable X.

$\boxed{\sigma^2=E[(X-\mu)^2]}$ and σ is called standard deviation.

We have;

$$\sigma^2=Var(X)=E[(X-\mu)^2]=E(X^2-2X\mu+\mu^2)$$

$$=E(X^2)-2\mu.\underbrace{E(X)}_{=\mu}+\mu^2=E(X^2)-2\mu^2+\mu^2$$

$$\boxed{\sigma^2=E(X^2)-\mu^2}$$

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1.89. Find the mean and variance, if they exist, of each of the following distributions.

(a) $f(x) = \dfrac{3!}{x!\,(3-x)!}\left(\dfrac{1}{2}\right)^3$, $x = 0, 1, 2, 3$, zero elsewhere.

(b) $f(x) = 6x(1-x)$, $0 < x < 1$, zero elsewhere.

(c) $f(x) = 2/x^3$, $1 < x < \infty$, zero elsewhere.

1.89) a) $f(x) = \dfrac{3!}{x!\,(3-x)!}\left(\dfrac{1}{2}\right)^3 \quad x = 0, 1, 2, 3$

$f(0) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$; $f(1) = 3.\left(\frac{1}{2}\right)^3 = \frac{3}{8}$; $f(2) = 3.\left(\frac{1}{2}\right)^3 = \frac{3}{8}$; $f(3) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$$\mu = E(X) = \sum_{x=0}^{3} x.f(x) = 0.\frac{1}{8} + 1.\frac{3}{8} + 2.\frac{3}{8} + 3.\frac{1}{8} = \frac{12}{8} = \frac{3}{2}$$

$$E(X^2) = \sum_{x=0}^{3} x^2.f(x) = 0^2.\frac{1}{8} + 1^2.\frac{3}{8} + 2^2.\frac{3}{8} + 3^2.\frac{1}{8} = \frac{24}{8} = 3$$

$$\sigma^2 = Var(X) = E(X^2) - \mu^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$$

b) $f(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & o.w \end{cases}$

$$\mu = E(X) = \int_{-\infty}^{\infty} x.f(x)\,dx = \int_0^1 x.6x.(1-x)\,dx = 6\int_0^1 (x^2 - x^3)\,dx$$

$$= 6.\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6.\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

$$E(X^2) = \int_0^1 x^2.6x(1-x)\,dx = 6\int_0^1 (x^3 - x^4)\,dx = 6.\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{6}{20} = \frac{3}{10}$$

$$\sigma^2 = Var(X) = E(X^2) - \mu^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{2}{40} = \frac{1}{20}$$

c) Note that, $E(|X|) = E(X)$ since X is defined for positive values. So, we do NOT need to check if $\int_{-\infty}^{\infty} |x| f(x)dx$ converges to a finite limit.

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_{1}^{\infty} x \cdot \frac{2}{x^3} dx = 2\int_{1}^{\infty} \frac{1}{x^2} dx$$

$$= -2 \cdot \lim_{k\to\infty} \left(\frac{1}{x}\right]_1^k = -2 \cdot \lim_{k\to\infty} \left(\frac{1}{k} - 1\right) = -2 \cdot -1 = 2$$

$$E(X^2) = \int_{1}^{\infty} x^2 \cdot \frac{2}{x^3} dx = 2\int_{1}^{\infty} \frac{1}{x} dx = 2 \lim_{k\to\infty} \ln(x)\Big]_1^k$$

$$= 2 \cdot \lim_{k\to\infty} (k-1) \to \infty$$

Then, $Var(X) \to \infty$ (does NOT exist)

Example: let $f(x) = \binom{n}{x} \cdot p^x (1-p)^{n-x}$ $\quad x = 0, 1, 2, \ldots, n$

Find $E(X)$

Answer: $E(X) = \sum_{x=0}^{n} x \cdot \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} = \sum_{x=1}^{n} \underbrace{x \cdot \binom{n}{x}} p^x (1-p)^{n-x}$

$$= x \cdot \frac{n!}{x!(n-x)!} = \frac{x \cdot n \cdot (n-1)!}{x \cdot (x-1)! \cdot (n-x)!} = n \cdot \binom{n-1}{x-1}$$

$$= \sum_{x=1}^{n} n \cdot \binom{n-1}{x-1} \cdot p \cdot p^{x-1} (1-p)^{n-x}$$

$$= n \cdot p \cdot \sum_{x=1}^{n} \binom{n-1}{x-1} \cdot p^{x-1} (1-p)^{n-x} = n \cdot p \cdot \underbrace{\sum_{x=0}^{n-1} \binom{n-1}{x} p^x (1-p)^{n-1-x}}_{=1}$$

$$= np$$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

Moment Generating Function (mgf)

* Remember;

$$E[g(x)] = \begin{cases} \sum_x g(x) f(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f(x) dx & \text{if } X \text{ is continuous.} \end{cases}$$

Now, consider the function $g(t) = e^{tX}$

Its derivatives with respect to t are;

$$g^{(1)}(t) = \frac{dg(t)}{dt} = \frac{d\, e^{tx}}{dt} = X \cdot e^{tX}$$

$$g^{(2)}(t) = \frac{d^2 g(t)}{dt^2} = \frac{d}{dt}(X \cdot e^{tx}) = X^2 \cdot e^{tX}$$

and in general, $g^{(k)}(t) = X^k \cdot e^{tX}$

* The expectation $E(X^k)$ is called "k^{th} moment of X"

k^{th} derivative

Let $M^{(k)}(t) = E[g^{(k)}(t)] = E[X^k \cdot e^{tX}]$

Then, $M(t)$ is called "Moment Generating Function"

because $\boxed{M^{(k)}(0) = E(X^k)}$ $\boxed{M(t) = E(e^{tX})}$

* Note that; $\mu = E(X) = M^{(1)}(0)$ and

$$\sigma^2 = E(X^2) - \mu^2 = M^{(2)}(0) - [M^{(1)}(0)]^2$$

Moment Generating Function has the following properties:

(i) If X has mgf $M_X(t)$ and $Y = aX + b$ then

$$\boxed{M_Y(t) = e^{tb} \cdot M_X(at)}$$ because

$$M_Y(t) = E(e^{Yt}) = E(e^{(aX+b)t}) = E(e^{aXt} \cdot e^{bt})$$
$$= e^{bt} \cdot E(e^{at.X}) = e^{bt} \cdot m(at)$$

(ii) Let $Y = X_1 + X_2 + \dots + X_n$ and X_i are independent.

Then $$\boxed{M_Y(t) = M_1(t) . M_2(t) . \dots . M_n(t)}$$

Noting that if X and Y are independent random variables then $E[g(X).h(Y)] = E[g(X)].E[h(Y)]$, we have;

$$M_Y(t) = E(e^{Yt}) = E(e^{(X_1 + X_2 + \dots + X_n).t})$$
$$= E(e^{X_1 t} \cdot e^{X_2 t} \cdot \dots \cdot e^{X_n t}) = E(e^{X_1 t}). E(e^{X_2 t}) \dots . E(e^{X_n t})$$
$$= M_1(t). M_2(t). \dots . M_n(t)$$

Example: Find mgf for the Bernoulli random variable.

Answer: $X \sim Bernoulli(p)$ → Number of success on a single trial when prob. of success is p.

$$f(x) = p^x (1-p)^{1-x}, \quad x \in \{0, 1\}$$

$$m(t) = E(e^{tX}) = \sum_{x=0}^{1} e^{tx} \cdot f(x) = e^{t.0} \cdot f(0) + e^{t.1} \cdot f(1)$$

$$m(t) = (1-p) + e^t . p = p.e^t + (1-p)$$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

Example: Find the mgf of Binomial random variable and use mgf to find its mean and variance

Answer: Let X_i are iid (independent and identically distributed) Bernoulli trials for $i=1,2,\dots,n$

Then, if $Y = X_1 + X_2 + \dots + X_n$

$Y \sim Binomial(n;p)$ → Number of success on n independent trials with constant probability of success.

$f(y) = \binom{n}{y} p^y (1-p)^{n-y}$ $\quad y=0,1,\dots,n$

We have,

$M_y(t) = M_1(t).M_2(t).\dots.M_n(t) = [M(t)]^n = [p.e^t + (1-p)]^n$

$M_y^{(1)}(t) = n.p.e^t[p.e^t + (1-p)]^{n-1}$

$* \mu = M_y^{(1)}(0) = E(Y) = n.p.[p+1-p]^{n-1} = n.p$

$M_y^{(2)}(t) = n.p.e^t.(n-1).p.[p.e^t + (1-p)]^{n-2} + n.p.e^t.[p.e^t + (1-p)]^{n-1}$

$M_y^{(2)}(0) = E(Y^2) = np^2(n-1) + np$

$* \sigma^2 = E(Y^2) - \mu^2 = n^2p^2 - np^2 + np - n^2p^2 = np - np^2 = np(1-p)$

Example: Find mgf of Poisson random variable and use mgf to find its mean and variance

Answer: $X \sim Poisson(\lambda) \Rightarrow f(x) = \frac{e^{-\lambda}.\lambda^x}{x!}$ $\quad x=0,1,2,\dots$

$m(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}.\frac{e^{-\lambda}.\lambda^x}{x!} = e^{-\lambda}.\underbrace{\sum_{x=0}^{\infty} \frac{[e^t.\lambda]^x}{x!}}_{=e^{\lambda.e^t}}$

$= e^{-\lambda}.e^{\lambda.e^t} = e^{\lambda[e^t-1]}$

$$m^{(1)}(t) = \lambda . e^t . e^{\lambda[e^t - 1]} \quad ; m^{(2)}(t) = \lambda^2 . e^t . e^t . e^{\lambda[e^t-1]} + \lambda . e^t . e^{\lambda[e^t-1]}$$

$$\mu = E(X) = m^{(1)}(0) = \lambda \quad ; E(X^2) = m^{(2)}(0) = \lambda^2 + \lambda$$

$$\sigma^2 = Var(X) = E(X^2) - \mu^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

Example: let X: time to complete a certain task has an Exponential Distribution with mean 10. Also let, C: Cost of completing this task is given by;

$$C = 100 + 40X + 3X^2$$

Find the mean and variance of C.

Answer: $X \sim \text{Exponential}(\theta)$

$$f(x) = \frac{1}{\theta} \cdot e^{-x/\theta}$$

$$m(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} . f(x)\, dx = \int_0^{\infty} e^{tx} . \frac{1}{\theta} \cdot e^{-x/\theta}\, dx$$

$$= \frac{1}{\theta} \cdot \int_0^{\infty} e^{-x(\frac{1}{\theta} - t)}\, dx = \frac{1}{\theta} \cdot \int_0^{\infty} e^{-x(1-\theta t)/\theta}\, dx$$

let $\frac{x(1-\theta t)}{\theta} = u$

$\frac{(1-\theta t)}{\theta} dx = du$

$dx = \frac{\theta}{1-\theta t} du$

$$m(t) = \frac{1}{\theta} \cdot \frac{\theta}{1-\theta t} \cdot \underbrace{\int_0^{\infty} e^{-u}\, du}_{=1 \text{ (show this)}}$$

$$\underline{m(t) = (1-\theta t)^{-1}}$$

$$m^{(1)}(t) = -1.(-\theta).(1-\theta t)^{-2} = \theta . (1-\theta t)^{-2} \; ; \; m^{(2)}(t) = -2.\theta.(-\theta).(1-\theta t)^{-3} = 2\theta^2 . (1-\theta t)^{-3}$$

Likewise, $m_3(t) = 3!.\theta^3 . (1-\theta t)^{-4}$ and $m^{(4)}(t) = 4!\,\theta^4 . (1-\theta t)^{-5}$

So; $E(X) = \theta$; $E(X^2) = 2\theta^2$; $E(X^3) = 6\theta^3$ and $E(X^4) = 24\theta^4$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

$X \sim \text{Exponential}(\theta=10)$

$C = 100 + 40X + 3X^2$

$E(C) = E(100+40X+3X^2) = 100 + 40E(X) + 3E(X^2)$
$= 100 + 40.10 + 3.2.10^2 = \underline{1100}$

$Var(C) = Var(100+40X+3X^2) = Var(40X+3X^2)$
$= E[(40X+3X^2)^2] - E^2(40X+3X^2)$

$E(40X+3X^2) = 40E(X) + 3E(X^2) = 40.10 + 3.2.10^2 = 1000$

$E^2(40X+3X^2) = 1000^2 = 1000000$

$E[(40X+3X^2)^2] = E[1600X^2 + 240X^3 + 9X^4]$
$= 1600E(X^2) + 240E(X^3) + 9E(X^4)$
$= 1600\,2.10^2 + 240.6.10^3 + 9.24.10^4 = 3920000$

Then; $Var(C) = 3920000 - 1000000 = \underline{2920000}$

Example: Find Mean and Variance of Exponential Random Variable. Also find Skewness and Kurtosis.

Answer: We have shown that for an Exponential Random Variable with parameter θ; $E(X^k) = k!.\theta^k$

Then, $\mu = E(X) = \theta$; $E(X^2) = 2\theta^2$; $\sigma^2 = Var(X) = 2\theta^2 - \theta^2 = \theta^2$

Skewness is defined as $\underbrace{SK = \frac{E[(X-\mu)^3]}{\sigma^3}}$ → Measures Symmetry and

Kurtosis is defined as $\underbrace{KT = \frac{E[(X-\mu)^4]}{\sigma^4}}$ → Measures Peakness

$$E[(X-\mu)^3] = E(X^3 - 3X^2\mu + 3X\mu^2 - \mu^3)$$
$$= E(X^3) - 3\mu E(X^2) + 3\mu^2 E(X) - \mu^3$$
$$= 6\theta^3 - 3\theta \cdot 2\theta^2 + 3\theta^2 \cdot \theta - \theta^3 = 2\theta^3$$

$$E[(X-\mu)^4] = E(X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4)$$
$$= E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 4\mu^3 E(X) + \mu^4$$
$$= 24\theta^4 - 4\theta \cdot 6\theta^3 + 6\theta^2 \cdot 2\theta^2 - 4\theta^3 \cdot \theta + \theta^4 = 9\theta^4$$

Then; $SK = \dfrac{2\theta^3}{\theta^3} = 2$ and $KT = \dfrac{9\theta^4}{\theta^4} = 9$

Example: Find mgf of gamma distribution with parameters α and β.

Answer: $X \sim Gamma(\alpha; \beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta} \quad x \geq 0$$

$$m(t) = E(e^{tX}) = \int_0^\infty e^{tx} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha) \cdot \beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-x\left(\frac{1}{\beta} - t\right)} dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} \cdot e^{-\frac{x}{\left(\frac{1}{\frac{1-\beta t}{\beta}}\right)}} dx$$

$$m(t) = \frac{1}{\beta^\alpha} \cdot (\beta')^\alpha \cdot \underbrace{\int_0^\infty \frac{1}{\Gamma(\alpha) \cdot (\beta')^\alpha} \cdot x^{\alpha-1} \cdot e^{-x/\beta'} dx}_{=1 \text{ because } \int_{-\infty}^{\infty} f(x)dx = 1} \quad \text{let } \frac{1}{\frac{1-\beta t}{\beta}} = \beta'$$

$$m(t) = \frac{1}{\beta^\alpha} \cdot \frac{1}{\left(\frac{1-\beta t}{\beta}\right)^\alpha} = \frac{1}{\beta^\alpha} \cdot \frac{\beta^\alpha}{(1-\beta t)^\alpha} = (1-\beta t)^{-\alpha}$$

(iii) Moment Generating function is unique and completely determines the distribution of the random variable. Namely, if two random variables have the same mgf, they have (exactly) the same distribution.

Example: Let mgf of a random variable X is given by $M(t) = \frac{4}{10}e^{t} + \frac{3}{10}e^{2t} + \frac{1}{10}e^{3t} + \frac{2}{10}e^{4t}$. Find pmf of this distribution.

Answer: We have; $M(t) = E(e^{tX}) = \sum_x e^{tx} \cdot f(x)$

then; $\frac{4}{10} \cdot e^{t} + \frac{3}{10} \cdot e^{2t} + \frac{1}{10} \cdot e^{3t} + \frac{2}{10} \cdot e^{4t}$

$= f(x_1) \cdot e^{x_1 \cdot t} + f(x_2) \cdot e^{x_2 t} + f(x_3) \cdot e^{x_3 t} + f(x_4) \cdot e^{x_4 t}$

and so;

X	1	2	3	4
$f(x)$	$4/10$	$3/10$	$1/10$	$2/10$

Example: Let $X_i \overset{i.i.d}{\sim} Normal(\mu, \sigma^2)$ is a random sample $i = 1, 2, \ldots, n$. Find the distribution of sample mean $\bar{X} = \frac{\sum X_i}{n}$, if $M(t) = e^{\left(\frac{\sigma^2 t^2}{2} + \mu t\right)}$

Answer: $X \sim Normal(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Let $Y = X_1 + X_2 + \ldots + X_n = \sum X_i$ and $\bar{X} = \frac{Y}{n}$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

By property (ii);

$$M_y(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \ldots \cdot M_{X_n}(t) = [m(t)]^n = e^{\frac{n\sigma^2 t^2}{2} + n\mu t}$$

By property (i) with $a = \frac{1}{n}$ and $b = 0$;

$$m_{\bar{x}}(t) = m_y\left(\frac{1}{n}t\right) = e^{\frac{\sigma^2 t^2}{2n} + \mu t} = e^{\frac{(\sigma^2/n) \cdot t^2}{2} + \mu t}$$

By property (iii);

$$\bar{X} \sim Normal\left(\mu ; \frac{\sigma^2}{n}\right)$$

1.90. Let $f(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, \ldots$, zero elsewhere, be the p.d.f. of the random variable X. Find the m.g.f., the mean, and the variance of X.

1.90) $m(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} \cdot \left(\frac{1}{2}\right)^x = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$

$$= \sum_{x=0}^{\infty} \left(\frac{e^t}{2}\right)^x - 1 = \frac{1}{1 - \frac{e^t}{2}} - 1 = \frac{2}{2 - e^t} - 1 = \frac{e^t}{2 - e^t} \quad \left(\text{for } \frac{e^t}{2} < 1\right)$$

$$m'(t) = \frac{e^t(2 - e^t) + e^t \cdot e^t}{(2 - e^t)^2} = \frac{2e^t}{(2 - e^t)^2} \Rightarrow E(X) = m'(0) = 2 = \mu$$

$$m''(t) = \frac{2e^t \cdot (2 - e^t)^2 + 2(2 - e^t) \cdot 2e^{2t}}{(2 - e^t)^4} = \frac{2e^t(2 - e^t)(2 + e^t)}{(2 - e^t)^4}$$

$$\Rightarrow E(X^2) = m''(0) = \frac{2 \cdot (2-1)(2+1)}{(2-1)^4} = 6$$

$$\sigma^2 = Var(X) = E(X^2) - \mu^2 = 6 - 2^2 = 2$$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com