ECONOMETRICS-I Lecture Notes | Chapters 7 & 8

MULTIPLE REGRESSION ANALYSIS: ESTIMATION

Model (PRM): $Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + --- + \beta_k X_{ki}$

Consider the Three Variable Regression Model;

$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + u_i$ (for simplicity)

Like in Simple Regression, We have the following assumptions. Note that some of these assumptions are NEW!

(i) Zero mean value of u_i : $E(u_i / X_{2i}, X_{3i}) = 0$

(ii) No Serial Correlation : $Cov(u_i, u_j) = 0$ for $i \neq j$

(iii) Homoscedasticity : $E(u_i^2) = Var(u_i) = \sigma^2$

(iv) u_i and Each X_j are uncorrelated: $Cov(u_i, X_{2i}) = Cov(u_i, X_{3i}) = 0$

(v) No exact collinearity between X_2 and X_3 (or in general, any X_j can NOT be written as linear combination of others)

Example: Consider the following model:

$GNP_t = \beta_1 + \beta_2 M_t + \beta_3 M_{t-1} + \beta_4 (M_t - M_{t-1}) + u_t$

a) Assuming that you have the data to estimate this model, would you succeed in estimating all the coefficients?

b) If NOT, which coefficients can be estimated!

c) Suppose that $\beta_3 M_{t-1}$ is absent from the model. Answer (a) again.

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Answer4 a) Let $X_2 = M_t$; $X_3 = M_{t-1}$ and $Y = GNP_t$

we have, $Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + \beta_4 X_{4t}$

where $X_{4t} = X_{2t} - X_{3t}$

Since X_4 is a linear Combination of X_2 and X_3, all parameters can NOT be estimated.

b) $Y = \beta_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 (X_2 - X_3)$

$Y = \beta_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_2 - \beta_4 X_3$

$Y = \beta_1 + (\beta_2 + \beta_4) X_2 + (\beta_3 - \beta_4) X_3$

If we omit β_4 from the model, β_2 and β_3 can be estimated. (or in general, 2 of 3 coefficients can be estimated.)

c) $Y = \beta_1 + \beta_2 X_2 + \beta_4 (X_2 - X_3)$

$Y = \beta_1 + \underbrace{(\beta_2 + \beta_4)}_{\psi_1} X_2 + \underbrace{\beta_4}_{\psi_2} (-X_3)$

$\beta_4 = \psi_2 \Rightarrow \beta_2 = \psi_1 - \psi_2$, both can be estimated

(Note that β_1 can always be estimated)

7.12. Consider the following models.*

$$\text{Model A:} \quad Y_t = \alpha_1 + \alpha_2 X_{2t} + \alpha_3 X_{3t} + u_{1t}$$

$$\text{Model B:} \quad (Y_t - X_{2t}) = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_{2t}$$

a. Will OLS estimates of α_1 and β_1 be the same? Why?
b. Will OLS estimates of α_3 and β_3 be the same? Why?
c. What is the relationship between α_2 and β_2?
d. Can you compare the R^2 terms of the two models? Why or why not?

7.12) Model B: $(Y_t - X_{2t}) = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_{2t}$

$$Y_t = \beta_1 + \beta_2 X_{2t} + X_{2t} + \beta_3 X_{3t} + u_{2t}$$

$$Y_t = \beta_1 + (\beta_2 + 1) X_{2t} + \beta_3 X_{3t} + u_{2t}$$

Model A: $Y_t = \alpha_1 + \alpha_2 X_{2t} + \alpha_3 X_{3t} + u_{1t}$

a) Yes b) Yes c) $\alpha_2 = \beta_2 + 1$

d) The two R^2 values will be exactly the same.

Matrix Notation of Linear Regression Model:

PRF: $Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \dots + \beta_k X_{ki} + u_i \quad i = 1, 2, \dots, n$

This can be written as a set of n simultaneous equations;

$$Y_1 = \beta_1 + \beta_2 X_{21} + \beta_3 X_{31} + \dots + \beta_k X_{k1} + u_1$$
$$Y_2 = \beta_1 + \beta_2 X_{22} + \beta_3 X_{32} + \dots + \beta_k X_{k2} + u_2$$
$$\dots$$
$$Y_n = \beta_1 + \beta_2 X_{2n} + \beta_3 X_{3n} + \dots + \beta_k X_{kn} + u_n$$

Then, we form the following Matrices;

$$\underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}}_{\substack{Y \\ (n\times 1)}} = \underbrace{\begin{bmatrix} 1 & X_{21} & X_{31} & \dots & X_{k1} \\ 1 & X_{22} & X_{32} & \dots & X_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & X_{3n} & \dots & X_{kn} \end{bmatrix}}_{\substack{X \\ (n\times k)}} \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}}_{\substack{\beta \\ (k\times 1)}} + \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}}_{\substack{u \\ (n\times 1)}}$$

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$$\boxed{\underset{(n\times 1)}{y} = \underset{(n\times k)}{X}\,\underset{(k\times 1)}{\beta} + \underset{(n\times 1)}{u}}$$

→ Matrix Representation of Regression function.

Assumptions:

(i) $$E(\underline{u}) = E\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} E(u_1) \\ E(u_2) \\ \vdots \\ E(u_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}$$ → shows that this is a matrix

$$\boxed{E(\underline{u}) = \underline{0}}$$

(ii)-(iii) $$Var(\underline{u}) = E(\underline{u}\,\underline{u}') = E\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

→ transpose of u

$$= E\begin{bmatrix} u_1^2 & u_1 u_2 & \cdots & u_1 u_n \\ u_2 u_1 & u_2^2 & \cdots & u_2 u_n \\ \cdots & \cdots & & \\ u_n u_1 & u_n u_2 & \cdots & u_n^2 \end{bmatrix} = \begin{bmatrix} E(u_1^2) & E(u_1 u_2) & \cdots & E(u_1 u_n) \\ E(u_2 u_1) & E(u_2^2) & \cdots & E(u_2 u_n) \\ \cdots & \cdots & & \\ E(u_n u_1) & E(u_n u_2) & \cdots & E(u_n^2) \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \sigma^2 I$$

$$\boxed{Var(\underline{u}) = E(\underline{u}\,\underline{u}') = \sigma^2 I}$$

(v) No multicollinearity ⇒ $\lambda_1 X_{1i} + \lambda_2 X_{2i} + \cdots + \lambda_k X_{ki} = 0$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}$$

$$\boxed{\underset{(1\times k)}{\lambda'}\;\underset{(k\times 1)}{x} = \underset{\text{scaler}}{0}}$$

OLS Estimation

Population Model: $\underset{(n\times 1)}{y} = \underset{(n\times k)}{X}\,\underset{(k\times 1)}{\beta} + \underset{(n\times 1)}{u}$

Sample Model: $\underset{(n\times 1)}{y} = \underset{(n\times k)}{X}\,\underset{(k\times 1)}{\hat{\beta}} + \underset{(n\times 1)}{\hat{u}}$

Note that;

$$\underline{\hat{u}}'\underline{\hat{u}} = [\hat{u}_1 \; \hat{u}_2 \; \cdots \; \hat{u}_n] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_n^2 = \sum \hat{u}_i^2$$

we want to minimize $\underline{\hat{u}}'\underline{\hat{u}}$ with respect to $\underline{\hat{\beta}}$

$$\underline{\hat{u}} = \underline{y} - \underline{X}\,\underline{\hat{\beta}}$$

Note that; $(AB)' = B'A'$

All Are matrices.

$$\hat{u}'u = (y - X\hat{\beta})'(y - X\hat{\beta}) = (y' - \hat{\beta}'X')(y - X\hat{\beta})$$

$$= y'y - \underbrace{y'X\hat{\beta}}_{=\hat{\beta}'X'y} - \hat{\beta}'X'y + \hat{\beta}'X'X\hat{\beta} = y'y - 2\hat{\beta}'X'y + \hat{\beta}'X'X\beta$$

(since scaler, $A' = A$)

like $(\cancel{x}\beta)^2$ for scaler case

$$\hat{u}'u = y'y - 2\hat{\beta}'X'y + \hat{\beta}'X'X\beta$$

like $2\cancel{x}^2\beta$

$$\frac{\partial \hat{u}'u}{\partial \hat{\beta}} = -2X'y + 2(X'X)\hat{\beta} = 0$$

$$(X'X).\hat{\beta} = X'y$$

$$\boxed{\hat{\beta} = (X'X)^{-1}.\,X'y}$$

2, Normal Equations.

Note that;

$$X' = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{k1} & X_{k2} & \cdots & X_{kn} \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 1 & X_{21} & \cdots & X_{k1} \\ 1 & X_{22} & \cdots & X_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & \cdots & X_{kn} \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$X'X = \begin{array}{c} \\ 1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & x_2 & x_3 & \cdots & x_k \end{array} \\ \begin{bmatrix} n & \sum X_{2i} & \sum X_{3i} & \cdots & \sum X_{ki} \\ \sum X_{2i} & \sum X_{2i}^2 & \sum X_{2i}X_{3i} & \cdots & \sum X_{2i}X_{ki} \\ \sum X_{3i} & \sum X_{3i}X_{2i} & \sum X_{3i}^2 & \cdots & \sum X_{3i}X_{ki} \\ - & - & - & - & \\ \sum X_{ki} & \sum X_{ki}X_{2i} & \sum X_{ki}X_{3i} & \cdots & \sum X_{ki}^2 \end{bmatrix} \end{array} \qquad X'y = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \\ \vdots \\ x_k \end{array} \begin{array}{c} y \\ \begin{bmatrix} \sum y_i \\ \sum X_{1i}y_i \\ \sum X_{2i}y_i \\ \vdots \\ \sum X_{ki}y_i \end{bmatrix} \end{array}$$

and so; $(X'X)\hat{\beta} = X'y$ in scaler form it;

Normal Equations

$$\left\{ \begin{array}{l} n\hat{\beta}_1 + \hat{\beta}_2 \sum X_{2i} + \hat{\beta}_3 \sum X_{3i} + \cdots + \hat{\beta}_k \sum X_{ki} = \sum y_i \\ \hat{\beta}_1 \sum X_{2i} + \hat{\beta}_2 \sum X_{2i}^2 + \hat{\beta}_3 \sum X_{2i}X_{3i} + \cdots + \hat{\beta}_k \sum X_{2i}X_{ki} = \sum X_{2i}y_i \\ \hat{\beta}_1 \sum X_{3i} + \hat{\beta}_2 \sum X_{3i}X_{2i} + \hat{\beta}_3 \sum X_{3i}^2 + \cdots + \hat{\beta}_k \sum X_{3i}X_{ki} = \sum X_{3i}y_i \\ - - - - \\ \hat{\beta}_1 \sum X_{ki} + \hat{\beta}_2 \sum X_{ki}X_{2i} + \hat{\beta}_3 \sum X_{ki}X_{3i} + \cdots + \hat{\beta}_k \sum X_{ki}^2 = \sum X_{ki}y_i \end{array} \right.$$

* Note that in deviation form; $x_{ji} = X_{ji} - \bar{X}_j$ and $y_i = y_i - \bar{y}$;

we have $\underset{(n-1)\times 1}{\hat{\beta}} = \begin{bmatrix} \hat{\beta}_2 \\ \hat{\beta}_3 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}$ is found by deleting first row and column, and finally;

$$\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{X}_2 - \cdots - \hat{\beta}_k \bar{X}_k$$

Base Question: Given the following data in matrix form, answer the questions below:

$$Y = \begin{bmatrix} 6 \\ 2 \\ 16 \\ 6 \\ 10 \end{bmatrix} \qquad X = \begin{pmatrix} 1 & 6 & 10 \\ 1 & 2 & 8 \\ 1 & 10 & 12 \\ 1 & 4 & 8 \\ 1 & 8 & 12 \end{pmatrix}$$

a. Estimate the least squares regression line.
b. Compute ESS and TSS.
c. Compute and interpret coefficient of determination.
d. Estimate the error variance.
e. Estimate Var-Cov matrix.
f. Test the validity of the model.
g. Test the null hypothesis Ho: $\beta_2 = 0$
h. Test the null hypothesis Ho: $\beta_2 = \beta_3$
i. Test the null hypothesis Ho: $\beta_2 + \beta_3 \geq 1.5$

Base Question Answer - a) $\boxed{\hat{\beta} = (X'X)^{-1} \cdot (X'y)}$

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & 2 & 10 & 4 & 8 \\ 10 & 8 & 12 & 8 & 12 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 & 10 \\ 1 & 2 & 8 \\ 1 & 10 & 12 \\ 1 & 4 & 8 \\ 1 & 8 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 30 & 50 \\ 30 & 220 & 324 \\ 50 & 324 & 516 \end{bmatrix}$$

Remember, invers matrix is found by $[A|I] \rightarrow [I|A^{-1}]$

Then;

$$\left[\begin{array}{ccc|ccc} 5 & 30 & 50 & 1 & 0 & 0 \\ 30 & 220 & 324 & 0 & 1 & 0 \\ 50 & 324 & 516 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{5}R_1]{\substack{-6R_1+R_2 \\ -10R_1+R_3}} \left[\begin{array}{ccc|ccc} 1 & 6 & 10 & 0{,}2 & 0 & 0 \\ 0 & 40 & 24 & -6 & 1 & 0 \\ 0 & 24 & 16 & -10 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{40}R_2]{-\frac{6}{40}R_2+R_1 \quad -\frac{24}{40}R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 6,4 & 1,1 & -0,15 & 0 \\ 0 & 1 & 0,6 & -0,15 & 0,025 & 0 \\ 0 & 0 & 1,6 & -6,4 & -0,6 & 1 \end{array}\right] \xrightarrow[\frac{1}{1,6}R_3]{-\frac{6,4}{1,6}R_3+R_1 \quad -\frac{0,6}{1,6}R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 26,7 & 2,25 & -4 \\ 0 & 1 & 0 & 2,25 & 0,25 & -0,375 \\ 0 & 0 & 1 & -4 & -0,375 & 0,625 \end{array}\right]$$

$$\underbrace{\quad\quad}_{I} \qquad \underbrace{\quad\quad\quad\quad\quad\quad}_{(X'X)^{-1}}$$

So;
$$(X'X)^{-1} = \begin{bmatrix} 26,7 & 2,25 & -4 \\ 2,25 & 0,125 & -0,375 \\ -4 & -0,375 & 0,625 \end{bmatrix}$$

$$X'y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & 2 & 10 & 4 & 8 \\ 10 & 8 & 12 & 8 & 12 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ 16 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 40 \\ 304 \\ 436 \end{bmatrix}$$

Finally;

$$\hat{\beta} = (X'X)^{-1} \cdot (X'y) = \begin{bmatrix} 26,7 & 2,25 & -4 \\ 2,25 & 0,125 & -0,375 \\ -4 & -0,375 & 0,625 \end{bmatrix} \begin{bmatrix} 40 \\ 304 \\ 436 \end{bmatrix} = \begin{bmatrix} 8 \\ 2,5 \\ -1,5 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix}$$

The SRF is;

$$\boxed{\hat{y} = 8 + 2,5X_2 - 1,5X_3}$$

Unbiasedness of $\hat{\beta}$

Example: Show that $\hat{\beta}$ is an unbiased estimator of β

Answer: We'll show that $E(\hat{\beta}) = \beta$

We have; $\hat{\beta} = (X'X)^{-1} \cdot X'y$

matrices

$y = X\beta + u \Rightarrow$ Recall that this is PRF

Then; $\hat{\beta} = (X'X)^{-1} \cdot X'(X\beta + u) = \underbrace{(X'X)^{-1} X'X}_{= A^{-1}A = I} \beta + (X'X)^{-1} X' u$

$$\hat{\beta} = \beta + (X'X)^{-1} \cdot X' u$$

$$E(\hat{\beta}) = E[\underbrace{\beta + (X'X)^{-1} \cdot X'}_{\text{constant}} u] = \beta + (X'X)^{-1} \cdot X' \underbrace{E(u)}_{=0}$$

$$\boxed{E(\hat{\beta}) = \beta}$$

Variance - Covariance Matrix of $\hat{\beta}$

Remember, for a scaler Random Variable W;

$Var(W) = E[(W - E(W))^2]$ and $Cov(W, W) = Var(W)$

In matrix notation, square replaces $A \cdot A'$ → means A^2 for scaler

The variance - Covariance Matrix of $\hat{\beta}$ is;

$$Var\text{-}Cov(\hat{\beta}) = \begin{array}{c} \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{array} \begin{array}{c} \begin{array}{cccc} \hat{\beta}_1 & \hat{\beta}_2 & \cdots & \hat{\beta}_k \end{array} \\ \begin{bmatrix} Var(\hat{\beta}_1) & Cov(\hat{\beta}_1, \hat{\beta}_2) & \cdots & Cov(\hat{\beta}_1, \hat{\beta}_k) \\ Cov(\hat{\beta}_2, \hat{\beta}_1) & Var(\hat{\beta}_2) & \cdots & Cov(\hat{\beta}_2, \hat{\beta}_k) \\ \cdots & \cdots & & \\ Cov(\hat{\beta}_k, \hat{\beta}_1) & Cov(\hat{\beta}_k, \hat{\beta}_2) & \cdots & Var(\hat{\beta}_k) \end{bmatrix} \end{array}$$

We obtain Var-Cov Matrix as follows;

Remember; PRF: $y = X\beta + u$

matrices

$$\hat{\beta} = (X'X)^{-1} X'y = (X'X)^{-1} X'(X\beta + u) = \underbrace{(X'X)^{-1} X'X}_{I} \beta + (X'X)^{-1} X' u$$

$$\hat{\beta} = \beta + (X'X)^{-1} X' u$$

$$\hat{\beta} - \beta = (X'X)^{-1} X' u$$

$$Var\text{-}Cov(\hat{\beta}) = E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)']$$

$$= E\{[(X'X)^{-1}X'u][(X'X)^{-1}X'u]'\}$$

$$= E[\underbrace{(X'X)^{-1}X'}_{constant}uu'\underbrace{X(X'X)^{-1}}_{constant}]$$

↳ Square Matrix, Transpose is same with itself

$$= (X'X)^{-1} \cdot X' \underbrace{E(uu')}_{=\sigma^2 I} X(X'X)^{-1}$$

$$= \sigma^2 \underbrace{(X'X)^{-1}X'X}_{= A^{-1}A = A}(X'X)^{-1} = \sigma^2(X'X)^{-1}$$

$$\boxed{Var\text{-}Cov(\hat{\beta}) = \sigma^2 \cdot (X'X)^{-1}}$$

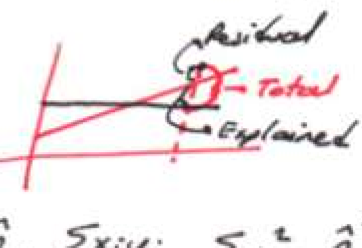
Note that;

$$\boxed{\hat{\sigma}^2 = \frac{\sum \hat{u}_i^2}{n-k}}$$

Coefficient of Determination.

Remember the formulas for two-variable case. We had;

$$TSS = ESS + RSS$$



where; $TSS = \sum y_i^2$

$$ESS = \hat{\beta}_2^2 \cdot \sum x_i^2 = \hat{\beta}_2 \cdot \hat{\beta}_2 \cdot \sum x_i^2 = \hat{\beta}_2 \cdot \frac{\sum x_i y_i}{\sum x_i^2} \cdot \sum x_i^2 = \hat{\beta}_2 \cdot \sum x_i y_i$$

$$RSS = \sum \hat{u}_i^2 = TSS - ESS = \sum y_i^2 - \hat{\beta}_2 \cdot \sum x_i y_i$$

Also Remember; $\sum y_i^2 = \sum Y_i^2 - \frac{(\sum Y_i)^2}{n} = \sum Y_i^2 - n \cdot \bar{Y}^2$

Note that, in matrix representation, $\underline{y}$ represents original values.

(NOT deviation from mean)

Likewise, for Multiple Regression, we have;

$$TSS = \sum y_i^2 = \underline{y}'\underline{y} - n.\bar{Y}^2$$

$$ESS = \hat{\beta}_2 \sum y_i x_{2i} + \hat{\beta}_3 \sum y_i x_{3i} + \dots + \hat{\beta}_k \sum y_i x_{ki} = \underline{\hat{\beta}}'\underline{X}'\underline{y} - n\bar{Y}^2$$

$$RSS = \sum \hat{u}_i^2 = \underline{\hat{u}}'\underline{u} = \overbrace{\underline{y}'\underline{y}}^{TSS} - \overbrace{\underline{\hat{\beta}}'\underline{X}'\underline{y}}^{ESS} \quad \left(\text{and } \hat{\sigma}^2 = \frac{RSS}{n-k}\right)$$

Then; $R^2 = \dfrac{ESS}{TSS} = \dfrac{\underline{\hat{\beta}}'\underline{X}'\underline{y} - n\bar{Y}^2}{\underline{y}'\underline{y} - n.\bar{Y}^2}$

$100.R^2$ part of the total variation in Y can be explained by the MODEL.

Base Question

Answer b) $\sum Y_i = 6+2+16+6+10 = 40$; $\bar{Y} = \frac{40}{5} = 8$; $n\bar{Y}^2 = 5.8^2 = 320$

$$TSS = \underline{y}'\underline{y} - n\bar{Y}^2 = \begin{bmatrix} 6 & 2 & 16 & 6 & 10 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 16 \\ 6 \\ 10 \end{bmatrix} - 5.8^2 = 432 - 320 = \underline{112}$$

$$ESS = \hat{\beta}'X'y - n\bar{Y}^2 = \begin{bmatrix} 8 & 2{,}5 & -1{,}5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & 2 & 10 & 4 & 8 \\ 10 & 8 & 12 & 8 & 12 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 16 \\ 6 \\ 10 \end{bmatrix} - 5.8^2$$

$$= 426 - 320 = \underline{106}$$

$(RSS = TSS - ESS = 112 - 106 = 6)$

c) $R^2 = \dfrac{ESS}{TSS} = \dfrac{106}{112} = 0{,}9464$

94,64% of the total variation in Y can be explained by the MODEL $\rightarrow$ X_2 and X_3

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d) $\hat{\sigma}^2 = \frac{RSS}{n-k} = \frac{6}{5-3} = \frac{6}{2} = 3$

e) $(X'X)^{-1} = \begin{bmatrix} 26,7 & 2,25 & -4 \\ 2,25 & 0,25 & -0,375 \\ -4 & -0,375 & 0,625 \end{bmatrix}$ and

$$\widehat{VarCov}(\hat{\beta}) = \hat{\sigma}^2 \cdot (X'X)^{-1} = 3 \cdot (X'X)^{-1} = \begin{array}{c c} & \begin{array}{ccc} \hat{\beta}_1 & \hat{\beta}_2 & \hat{\beta}_3 \end{array} \\ \begin{array}{c} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{array} & \begin{bmatrix} 80,1 & 6,75 & -12 \\ 6,75 & 0,75 & -1,125 \\ -12 & -1,125 & 1,875 \end{bmatrix} \end{array}$$

7.2. From the following data estimate the partial regression coefficients, their standard errors, and the adjusted and unadjusted R^2 values:

$$\bar{Y} = 367.693 \quad \bar{X}_2 = 402.760 \quad \bar{X}_3 = 8.0$$

$$\sum(Y_i - \bar{Y})^2 = 66042.269 \quad \sum(X_{2i} - \bar{X}_2)^2 = 84855.096$$

$$\sum(X_{3i} - \bar{X}_3)^2 = 280.000 \quad \sum(Y_i - \bar{Y})(X_{2i} - \bar{X}_2) = 74778.346$$

$$\sum(Y_i - \bar{Y})(X_{3i} - \bar{X}_3) = 4250.900 \quad \sum(X_{2i} - \bar{X}_2)(X_{3i} - \bar{X}_3) = 4796.000$$

$$n = 15$$

7.2) Note that, the given statistics are suitable for "Deviation Form." Namely;

$\bar{Y} = 367,639 \quad \bar{X}_2 = 402,760 \quad \bar{X}_3 = 8,0 \quad \boxed{n = 15}$

$\Sigma y_i^2 = 66042,269 \quad \Sigma x_{2i}^2 = 84855,096$

$\Sigma x_3^2 = 280 \quad \Sigma y_i x_{2i} = 74778,346$

$\Sigma y_i x_{3i} = 4250,9 \quad \Sigma x_{2i} x_{3i} = 4796$

$$y_i = \hat{\beta}_2 x_{2i} + \hat{\beta}_3 x_{3i} + \hat{u}_i \qquad \hat{\beta} = \begin{bmatrix} \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix}$$

$$x'x = \begin{bmatrix} \sum x_{2i}^2 & \sum x_{2i} x_{3i} \\ \sum x_{2i} x_{3i} & \sum x_{3i}^2 \end{bmatrix} = \begin{bmatrix} 84855{,}1 & 4796 \\ 4796 & 280 \end{bmatrix}$$

$$|x'x| = 84855{,}1 . 280 - 4796^2 = 757812$$

Remember; $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix}$

where $|A| = ad - bc$

So;

$$(x'x)^{-1} = \frac{1}{757812} \cdot \begin{bmatrix} 280 & -4796 \\ -4796 & 84855{,}1 \end{bmatrix} = \begin{bmatrix} 0{,}00037 & -0{,}00633 \\ -0{,}00633 & 0{,}112 \end{bmatrix}$$

$$(x'y) = \begin{bmatrix} \sum x_{2i} y_i \\ \sum x_{3i} y_i \end{bmatrix} = \begin{bmatrix} 74778{,}3 \\ 4250{,}9 \end{bmatrix}$$

$$\hat{\beta} = (x'x)^{-1}(x'y) = \begin{bmatrix} 0{,}00037 & -0{,}00633 \\ -0{,}00633 & 0{,}112 \end{bmatrix} \begin{bmatrix} 74778{,}3 \\ 4250{,}9 \end{bmatrix} = \begin{bmatrix} 0{,}727 \\ 2{,}736 \end{bmatrix} = \begin{bmatrix} \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix}$$

$$\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 . \bar{x}_2 - \hat{\beta}_3 . \bar{x}_3 = 367{,}6 - 0{,}727 . 402{,}8 - 2{,}736 . 8 = 53{,}106 = \hat{\beta}_1$$

$$\boxed{\hat{y} = 53{,}106 + 0{,}727 . X_{2i} + 2{,}736 . X_{3i}}$$

$$TSS = \sum y_i^2 = 66042{,}3$$

$$ESS = \hat{\beta}_2 . \sum y_i x_{2i} + \hat{\beta}_3 . \sum y_i x_{3i} = 0{,}727 . 74477\cancel{8}3 + 2{,}736 . 4250{,}9 = 65965{,}1$$

$$RSS = TSS - ESS = 66042{,}3 - 65965{,}1 = 77{,}2$$

$$R^2 = \frac{ESS}{TSS} = \frac{65965{,}1}{66042{,}3} = 0{,}9988 \qquad \boxed{R^2 = 0{,}9988}$$

Adjusted R^2 ($\bar{R}^2$)

Since R^2 shows the explained part by the model, R^2 always increases as we add new regressors to the model. However, having a model with too many regressors is NOT advised for some reasons we'll see later. So, we use "Adjusted R^2" especially when we compare two models. When we add a new regressor to a model, $\bar{R}$ increases only if it adds a sufficient explanation to the total variation.

7.2) $$\boxed{\bar{R}^2 = 1 - \frac{RSS/(n-k)}{TSS/(n-1)} = 1 - (1-R^2) \cdot \frac{n-1}{n-k}}$$

$$\bar{R}^2 = 1 - (1-0{,}9988) \cdot \frac{15-1}{15-3} = 0{,}9986$$

To Calculate Standard Errors, we need Var-Cov($\hat{\beta}$)

$$\hat{\sigma}^2 = \frac{RSS}{n-k} = \frac{77{,}2}{15-3} = 6{,}43$$

$$\text{Var-Cov}(\hat{\beta}) = \hat{\sigma}^2 \cdot (X'X)^{-1} = 6{,}43 \cdot (X'X)^{-1} = \begin{array}{c c} & \begin{array}{cc} \hat{\beta}_2 & \hat{\beta}_3 \end{array} \\ \begin{array}{c} \hat{\beta}_2 \\ \hat{\beta}_3 \end{array} & \begin{bmatrix} 0{,}002 & -0{,}041 \\ -0{,}041 & 0{,}720 \end{bmatrix} \end{array}$$

$$SE(\hat{\beta}_2) = \sqrt{\widehat{Var}(\hat{\beta}_2)} = \sqrt{0{,}002} = 0{,}045$$

$$SE(\hat{\beta}_3) = \sqrt{\widehat{Var}(\hat{\beta}_3)} = \sqrt{0{,}720} = 0{,}849$$

Finally; the model summary is;

$$\hat{y} = 53{,}106 + 0{,}727 X_{2i} + 2{,}736 X_{3i}$$
$$S.E. = \qquad (0{,}045) \qquad (0{,}849) \qquad \bar{R}^2 = 0{,}9986$$
$$n = 15$$