MATHEMATICAL STATISTICS_1 Lecture Notes | CHAPTER 5

Limiting Distributions

Convergence in Distribution

Let the distribution function $F_n(y)$ of the random variable Y_n depend upon n, $n = 1, 2, 3, \ldots$. If $F(y)$ is a distribution function and if $\lim_{n\to\infty} F_n(y) = F(y)$ for every point y at which $F(y)$ is continuous, then the sequence of random variables, $Y_1, Y_2, \ldots$, converges in distribution to a random variable with distribution function $F(y)$.

Example: Let $X \sim$ Uniform $(0;\theta)$ and Y_n is the n^{th} order statistics of a random sample $X_1, X_2, \ldots, X_n$. Find the limiting distribution of Y_n.

Answer: To find limiting distribution;

(i) Find $F_n(y)$

(ii) Find $F(y) = \lim_{n\to\infty} F_n(y)$

We have, $f(x) = \begin{cases} \frac{1}{\theta} & 0 < x < \theta \\ 0 & o.w \end{cases}$ $\quad F(x) = \frac{x}{\theta} \quad 0 < x < \theta$

$$g_n(y) = n.[F(y)]^{n-1}.f(y) = n.\left(\frac{y}{\theta}\right)^{n-1}.\frac{1}{\theta} = \frac{n.y^{n-1}}{\theta^n} \quad 0 < y < \theta$$

$$G_n(y) = \int_0^y g_n(w)dw = \int_0^y \frac{n.w^{n-1}}{\theta^n}.dw = \frac{n}{\theta^n}.\frac{w^n}{n}\Big]_0^y = \frac{y^n}{\theta^n} = \left(\frac{y}{\theta}\right)^n \quad 0 < y < \theta$$

$$\lim_{n\to\infty} G_n(y) = \begin{cases} 0 & y < \theta \\ 1 & y \geq \theta \end{cases}$$

Since $F(y) = \begin{cases} 0 & y < \theta \\ 1 & y \geq \theta \end{cases}$ is a distribution function AND $\lim_{n\to\infty} G_n(y) = F(y)$ at each point of continuity of $F(y)$, Y_n converges in distribution to a random variable that has a degenerate distribution at $y=\theta$.

Example: Let $X \sim N(\mu=0; \sigma^2=1^2)$ ⇒ standard Normal Distribution.
Also let $\bar{X}_n$ is the mean of the random sample $X_1, X_2, \ldots, X_n$. Find limiting distribution of $\bar{X}_n$

Answer: Note that $X \sim N(\mu; \sigma^2)$ has pdf

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ and } \bar{X} \sim N\left(\mu; \frac{\sigma^2}{n}\right)$$

So, pdf of Standard Normal Random Variable is;

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2} \text{ and } \bar{X}_n \text{ has pdf}$$

$$f(x) = \frac{1}{\sqrt{2\pi/n}} \cdot e^{-n\bar{x}^2/2} \text{ So, the cdf of } \bar{X} \text{ is;}$$

$$F_n(\bar{x}) = \int_{w=-\infty}^{w=\bar{x}} \frac{1}{\sqrt{2\pi/n}} \cdot e^{-nw^2/2}\, dw = \int_{-\infty}^{\sqrt{n}\bar{x}} \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-v^2/2}\, dv}_{V \sim N(0;1)}$$

$v = \sqrt{n}\, w$

$dv = \sqrt{n}\, dw$

$$\lim_{n\to\infty} F_n(\bar{x}) = \begin{cases} 0 & \bar{x} \le 0 \rightarrow \int_{-\infty}^{-\infty} f(x)\,dx \\ 1/2 & \bar{x} = 0 \rightarrow \text{half area of standard Normal R.V} \\ 1 & \bar{x} > 0 \rightarrow \int_{-\infty}^{\infty} f(x)\,dx \end{cases}$$

Now;

$$F(\bar{x}) = \begin{cases} 0 & \bar{x} < 0 \\ 1 & \bar{x} \ge 0 \end{cases} \quad \text{is a cdf and} \quad \lim_{n\to\infty} F_n(\bar{x}) = F(\bar{x})$$

at each point of continuity of $F(\bar{x})$ (note that $F(\bar{x})$ is NOT continuous at $\bar{x}=0$). So, the sequence $\bar{X}_1, \bar{X}_2, \bar{X}_3, \ldots$ converges in distribution to a random variable that has a degenerate distribution at $\bar{x}=0$

Example: let $X \sim$ Uniform $(0;\theta)$ and Y_n is the n^{th} order statistics of a random sample $X_1, X_2, \ldots, X_n$. Also let $Z_n = n.(\theta - Y_n)$. Find limiting distribution of Z_n

Answer: We found cdf of Y_n at our first example

$G_n(y) = \left(\frac{y}{\theta}\right)^n \quad 0 < y < \theta$ and $Z_n = n.(\theta - Y_n)$

$$\frac{Z_n}{n} = \theta - Y_n$$

$$Y_n = \theta - \frac{Z_n}{n} \text{ so; } 0 < y \le \theta$$

$$0 < \theta - \frac{z}{n} \le \theta$$

$$-\theta < -\frac{z}{n} \le 0$$

$$0 < z \le n\theta$$

$H_n(z) = P(Z_n \le z) = P(n.(\theta - Y_n) \le z)$

$= P\left(Y_n \ge -\frac{z}{n} + \theta\right) = 1 - P\left(Y_n \le \theta - \frac{z}{n}\right)$

$= 1 - G_n\left(\theta - \frac{z}{n}\right) = 1 - \left(\frac{\theta - \frac{z}{n}}{\theta}\right)^n$

$$H_n(z) = 1 - \left(1 - \frac{z}{n\theta}\right)^n \quad 0 < z \le n\theta$$

Remember; $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{bn} = e^{ab}$

So; $\lim_{n\to\infty} H_n(z) = \begin{cases} 0 & z \le 0 \\ 1-e^{-z/\theta} & z>0 \end{cases}$

Now;

$F(z) = \begin{cases} 0 & z \le 0 \\ 1-e^{-z/\theta} & z>0 \end{cases}$ is a cdf and $\lim_{n\to\infty} H_n(z) = F(z)$.

So, $F(z)$ is the limiting distribution of Z_n (which is NOT degenerate)

5.1. Let $\bar{X}_n$ denote the mean of a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Find the limiting distribution of $\bar{X}_n$. degenerate at μ

5.2. Let Y_1 denote the first order statistic of a random sample of size n from a distribution that has the p.d.f. $f(x) = e^{-(x-\theta)}$, $\theta < x < \infty$, zero elsewhere. Let $Z_n = n(Y_1 - \theta)$. Investigate the limiting distribution of Z_n.

5.3. Let Y_n denote the nth order statistic of a random sample from a distribution of the continuous type that has distribution function $F(x)$ and p.d.f. $f(x) = F'(x)$. Find the limiting distribution of $Z_n = n[1 - F(Y_n)]$.

5.1) $\bar{X}_n \sim N\left(\mu; \frac{\sigma^2}{n}\right)$

$f_n(\bar{x}) = \frac{1}{\sqrt{2\pi\frac{\sigma^2}{n}}} \cdot e^{-\frac{1}{2}\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right)^2}$

$F_n(\bar{x}) = \int_{-\infty}^{\bar{x}} \frac{1}{\sqrt{2\pi\sigma^2/n}} \cdot e^{-\frac{1}{2}\left(\frac{w-\mu}{\sigma/\sqrt{n}}\right)^2} dw$

$v = \frac{\sqrt{n}\cdot(w-\mu)}{\sigma}$

$dv = \frac{\sqrt{n}}{\sigma} dw$

$$F_n(\bar{x}) = \int_{-\infty}^{\sqrt{n}\cdot(\bar{x}-\mu)/\sigma} \underbrace{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}v^2}}_{V \sim N(0;1)} dv$$

so;

$$\lim_{n\to\infty} F_n(\bar{x}) = \begin{cases} 0 & \bar{x} < \mu \xrightarrow{} \int_{-\infty}^{-\infty} f(v)\,dv \\ 1/2 & \bar{x} = \mu \longrightarrow \text{half area at standard normal R.V} \\ 1 & \bar{x} > \mu \longrightarrow \int_{-\infty}^{\infty} f(v)\,dv \end{cases}$$

Like our previous example,

$$F(\bar{x}) = \begin{cases} 0 & \bar{x} < \mu \\ 1 & \bar{x} \geq \mu \end{cases}$$ is the limiting distribution of $F_n(\bar{x})$ which is degenerate at $\bar{x} = \mu$.

5.2) $f(x) = e^{-(x-\theta)} \quad x > \theta$

$$F(x) = \int_{\theta}^{x} e^{-(w-\theta)}\,dw = -e^{-(w-\theta)}\Big]_{\theta}^{x} = -e^{-(x-\theta)} - (-e^{0})$$

$$F(x) = 1 - e^{-(x-\theta)}$$

$$g_1(y) = n\cdot[1-F(y)]^{n-1}\cdot f(y) = n\cdot\left[1-1+e^{-(y-\theta)}\right]^{n-1}\cdot e^{-(y-\theta)}$$

$$g_1(y) = n\cdot e^{-n(y-\theta)}$$

$$G_1(y) = \int_{w=\theta}^{w=y} n\cdot e^{-n(w-\theta)}\,dw = \int_{0}^{n(y-\theta)} e^{-u}\,du = -e^{-u}\Big]_{0}^{n(y-\theta)}$$

$n\cdot(w-\theta) = u$

$n\,dw = du$

$$G_1(y) = 1 - e^{-n(y-\theta)} \quad y > \theta$$

$$Z_n = n.(y_1 - \theta) \qquad y > 0$$

$$y_1 = \theta + \frac{z_n}{n} \qquad \theta + \frac{z}{n} > 0$$

$$z > 0$$

$$H_n(z) = P(Z_n \leq z) = P\left(n.(y_1 - \theta) \leq z\right) = P\left(y_1 \leq \frac{z}{n} + \theta\right)$$

$$= G_1\left(\frac{z}{n} + \theta\right) = 1 - e^{-n\left(\frac{z}{n} + \theta - \theta\right)} = 1 - e^{-z} \quad z \geq 0$$

so, limiting distribution of Z_n is;

$$F(z) = \lim_{n \to \infty} H_n(z) = \begin{cases} 0 & z \leq 0 \\ 1 - e^{-z} & z > 0 \end{cases}$$

5.3) y_n: n^{th} order statistics of $X_1, \dots, X_n$ with $F(x)$

Then; $g_n(y) = n.[F(y)]^{n-1}.F'(y)$

$$G_n(y) = \int_{-\infty}^{y} n.[F(w)]^{n-1}.F'(w)\,dw = \int_{u_0}^{u_1} n.u^{n-1}\,du = \cancel{n}.\frac{u^n}{\cancel{n}}\Bigg]_{u_0}^{u_1}$$

$F(w) = u$

$F'(w)dw = du$

$$= [F(w)]^n\Bigg]_{-\infty}^{y}$$

$$G_n(y) = [F(y)]^n$$

we have; $Z_n = n.[1 - F(y_n)]$ $\Rightarrow$ Since $0 \leq F(y_n) \leq 1$

$$\frac{Z_n}{n} = 1 - F(y_n) \qquad \text{then} \quad 0 \leq 1 - \frac{z}{n} \leq 1$$

$$\boxed{0 \leq z \leq n}$$

$$H_n(z) = P(Z_n \leq z) = P\left(n.[1 - F(y_n)] \leq z\right) = P\left(1 - F(y_n) \leq \frac{z}{n}\right)$$

$$= P\left(F(y_n) \geq 1 - \frac{z}{n}\right) = 1 - P\left(F(y_n) \leq 1 - \frac{z}{n}\right)$$

$$= 1 - G_n\left[F^{-1}\left(1 - \frac{z}{n}\right)\right] = 1 - \left\{F\left[F^{-1}\left(1 - \frac{z}{n}\right)\right]\right\}^n = 1 - \left(1 - \frac{z}{n}\right)^n$$

$$H_n(z) = 1 - \left(1 - \frac{z}{n}\right)^n \quad 0 < z < n$$

$$F(z) = \lim_{n \to \infty} H_n(z) = \begin{cases} 0 & z \leq 0 \\ 1 - e^{-z} & z > 0 \end{cases}$$

is the limiting distribution of Z_n.

Chebyshev's Inequality

Let the random variable X have a distribution of probability about which we assume only that there is a finite variance σ^2. Then for every $k > 0$,

$$\boxed{P(|X - \mu| \geq k.\sigma) \leq \frac{1}{k^2}}$$

or equivalently, $P(|X - \mu| < k.\sigma) \geq 1 - \frac{1}{k^2}$

1.114. If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, use Chebyshev's inequality to determine a lower bound for the probability $\Pr(-2 < X < 8)$.

1.114) $\mu = E(X) = 3$; $\sigma^2 = E(X^2) - \mu^2 = 13 - 3^2 = 4$

$\sigma = \sqrt{4} = 2$

$(-2 < X < 8)$

$\mu - k.\sigma$ $\quad$ $\mu + k.\sigma \Rightarrow 3 + 2k = 8$

$2k = 5$

$k = 2,5$

$$P(-2 < X < 8) = P(\mu - 2,5\sigma < X < \mu + 2,5\sigma) = P(|X - \mu| < 2,5\sigma)$$

$$\geq 1 - \frac{1}{2,5^2} = 0,84$$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

Convergence in Probability

A sequence of random variables $X_1, X_2, X_3, \ldots$ converges in probability to a random variable X if for every $\epsilon > 0$,

$$\boxed{\lim_{n\to\infty} P(|X_n - X| < \epsilon) = 1}$$

or equivalently, $P(|X_n - X| \geq \epsilon) = 0$

Example: Let $\bar{X}_n$ denote the mean of a random sample of size n from a distribution that has mean μ and positive variance σ^2. Then; $\mu_{\bar{x}} = E(\bar{X}) = \mu$ and $\sigma_{\bar{x}}^2 = Var(\bar{X}) = \frac{\sigma^2}{n}$. Consider, for every $\epsilon > 0$, the probability

$$P(|\bar{X}_n - \mu| \geq \epsilon) = P\left(|\bar{X}_n - \mu| \geq k \cdot \sigma_{\bar{x}}\right)$$

where $k = \frac{\epsilon}{\sigma_{\bar{x}}} = \frac{\epsilon\sqrt{n}}{\sigma}$. By Chebyshev Inequality;

$$P(|\bar{X}_n - \mu| \geq k \cdot \sigma_{\bar{x}}) \leq \frac{1}{k^2} = \frac{\sigma^2}{n \cdot \epsilon^2}$$ Then,

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \epsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n \cdot \epsilon^2} = 0$$

So, $\bar{X}_n$, $n = 1, 2, 3, \ldots$ converges in probability to μ if σ^2 is finite. This result is called the weak law of large numbers.

The last example intuatively shows us a corollary.

A sequence of random variables Y_n converges in probability to μ ($Y_n \xrightarrow{P} \mu$) if and only if;

(i) $E(Y_n) = \mu_n$ where $\lim_{n\to\infty} \mu_n = \mu$

(ii) $Var(Y_n) = \sigma_n^2$ is finite for all n ($\sigma_n^2 < \infty$)

(iii) $\lim_{n\to\infty} Var(Y_n) = 0$

Chi Square (χ^2) distribution & Sample Variance

Let $Z_1, Z_2, \ldots, Z_n$ are standard normal random variables and let $W = \sum_{i=1}^{n} Z_i$. Then, W has a Chi-Square distribution with degrees of freedom n

$$W \sim \chi^2_{(n)}$$

$$E(W) = n \quad \text{and} \quad Var(W) = 2n$$

Chi-Square distribution is a special case of Gamma distribution where $\alpha = \frac{n}{2}$ and $\beta = 2$

Remember; $V \sim Gamma(\alpha; \beta)$

$$f(v) = \frac{1}{\Gamma(\alpha).\beta^{\alpha}} . v^{\alpha-1} . e^{-v/\beta} \quad v>0$$

$$E(V) = \alpha.\beta \quad \text{and} \quad Var(V) = \alpha\beta^2$$

and $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} . e^{-x} dx$ and $\Gamma(\alpha) = (\alpha-1)!$ when α is Natural #.

Let $X_1, X_2, \ldots, X_n$ is a random sample. The sample variance is defined as;

$$S^2 = \frac{\sum (X_i - \bar{X})^2}{n} \quad \text{or} \quad S^{*2} = \frac{\sum (X_i - \bar{X})^2}{n-1}$$

Note that $S^{*2} = \frac{n}{n-1} \cdot S^2$

If the sample is from $N(\mu; \sigma^2)$, we have

$$\frac{n S^2}{\sigma^2} = \frac{(n-1) S^{*2}}{\sigma^2} \sim \chi^2_{(n-1)}$$

Example: Let $Z_n \sim \chi^2_{(n)}$ and let $W_n = \frac{Z_n}{n^2}$. Show that $W_n \xrightarrow{P} 0$

Answer: From our corollary, we have:

(i) $E(Z_n) = n$ and $E(W_n) = E\left(\frac{Z_n}{n^2}\right) = \frac{1}{n^2} E(Z_n) = \frac{1}{n^2} \cdot n = \frac{1}{n}$

$$\lim_{n \to \infty} E(W_n) = \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0$$

(ii) $Var(W_n) = Var\left(\frac{Z_n}{n^2}\right) = \frac{1}{n^4} \cdot Var(Z_n) = \frac{1}{n^4} \cdot 2n = \frac{2}{n^3}$

$Var(W_n) = \frac{2}{n^3} < \infty$ for all n

(iii) $\lim_{n \to \infty} Var(W_n) = \lim_{n \to \infty} \left(\frac{2}{n^3}\right) = 0$

So, $W_n \xrightarrow{P} 0$.

Note that;
$E(aX+b) = a \cdot E(X)$ and
$Var(aX+b) = Var(aX) = a^2 Var(X)$

The following theorem relates a certain limiting distribution to convergence in probability to a constant.

Theorem: Let $F_n(y)$ denote the distribution function of a random variable Y_n whose distribution depends upon the positive integer n. The sequence $Y_1, Y_2, Y_3 ----$ converges in probability to the constant c if and only if the limiting distribution of Y_n is degenerate at c.

5.7. Let the random variable Y_n have a distribution that is $b(n, p)$.
(a) Prove that Y_n/n converges in probability to p. This result is one form of the weak law of large numbers.
(b) Prove that $1 - Y_n/n$ converges in probability to $1 - p$.

5.8. Let S_n^2 denote the variance of a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Prove that $nS_n^2/(n-1)$ converges in probability to σ^2.

5.9. Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0$, μ, and b are constants (not functions of n). Prove that W_n converges in probability to μ.
Hint: Use Chebyshev's inequality.

5.10. Let Y_n denote the nth order statistic of a random sample of size n from a uniform distribution on the interval $(0, \theta)$, as in Example 1 of Section 5.1. Prove that $Z_n = \sqrt{Y_n}$ converges in probability to $\sqrt{\theta}$.

5.7) $Y_n \sim$ Binomial $(n;p)$

$$E(Y_n) = np \quad \text{and} \quad Var(Y_n) = np(1-p)$$

a) Let $W_n = \frac{Y_n}{n}$; (i) $E(W_n) = E\left(\frac{Y_n}{n}\right) = \frac{1}{n}E(Y_n) = \frac{1}{n}\cdot np = p$ ✓

(ii) $Var(W_n) = Var\left(\frac{Y_n}{n}\right) = \frac{1}{n^2}Var(Y_n) = \frac{1}{n^2}n\cdot p\cdot(1-p) = \frac{p(1-p)}{n} < \infty$ ✓

(iii) $\lim_{n\to\infty} Var(W_n) = \lim_{n\to\infty}\left(\frac{p(1-p)}{n}\right) = 0$ ✓ So $W_n \xrightarrow{P} p$

b) Let $V_n = 1 - \frac{y_n}{n}$ (i) $E(V_n) = E\left(1 - \frac{y_n}{n}\right) = 1 - E\left(\frac{y_n}{n}\right) = 1 - p$ ✓

(ii) $Var(V_n) = Var\left(1 - \frac{y_n}{n}\right) = Var\left(\frac{y_n}{n}\right) = \frac{p(1-p)}{n} < \infty$ ✓

(iii) $\lim_{n \to \infty} Var(V_n) = \lim_{n \to \infty} \left(\frac{p(1-p)}{n}\right) = 0$ ✓ So $V_n \xrightarrow{p} 1-p$

5.8) $S_n^{*2} = \frac{n . S_n^2}{n-1}$ and let $T = \frac{(n-1) . S_n^{*2}}{\sigma^2} \sim \chi^2_{(n-1)}$

(i) $E(S_n^{*2}) = E\left[\frac{\sigma^2}{n-1} \cdot \frac{(n-1)S_n^{*2}}{\sigma^2}\right] = \frac{\sigma^2}{n-1} . E(T) = \frac{\sigma^2}{n-1}(n-1) = \sigma^2$ ✓

(ii) $Var(S_n^{*2}) = Var\left[\frac{\sigma^2}{n-1} \cdot \frac{(n-1)S_n^{*2}}{\sigma^2}\right] = \frac{\sigma^4}{(n-1)^2} . Var(T)$

$= \frac{\sigma^4}{(n-1)^2} . 2.(n-1) = \frac{2\sigma^4}{(n-1)} < \infty$ ✓

(iii) $\lim_{n \to \infty} Var(S_n^{*2}) = \lim_{n \to \infty} \left(\frac{2\sigma^4}{n-1}\right) = 0$ ✓

So; $S_n^{*} \xrightarrow{p} \sigma^2$

5.9) (i) $E(W_n) = \mu$ ✓ (ii) $Var(W_n) = \frac{b}{n^p} < \infty$ ✓

(iii) $\lim_{n \to \infty} Var(W_n) = \lim_{n \to \infty} \left(\frac{b}{n^p}\right) = 0$ ✓

So, $W_n \xrightarrow{p} \mu$

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

5.10) $X \sim \text{Uniform}(0;\theta) \Rightarrow F(x) = \frac{x}{\theta}$

Y_n : n^{th} order statistics of $X_1, X_2, \ldots, X_n$

$G_n(y) = \left(\frac{y}{\theta}\right)^n$ → See page 39 $y < \theta$

$Z_n = \sqrt{Y_n} \quad y < \theta$

$\sqrt{y} < \sqrt{\theta} \Rightarrow z < \sqrt{\theta}$

$H_n(z) = P(Z_n \le z) = P(\sqrt{Y_n} \le z) = P(Y_n \le z^2)$

$= G_n(z^2) = \left(\frac{z^2}{\theta}\right)^n \quad z < \sqrt{\theta}$

$$F(z) = \lim_{n\to\infty} H_n(z) = \begin{cases} 0 & z < \sqrt{\theta} \\ 1 & z \geq \sqrt{\theta} \end{cases}$$

Z_n is degenerate at $\sqrt{\theta}$. So, $Z_n \xrightarrow{P} \sqrt{\theta}$

Limiting Moment-Generating Functions

Let the random variable Y_n have the distribution function $F_n(y)$ and the mgf $M(t;n)$ that exists for $-h<t<h \ \forall n$. If there exists a distribution function $F(y)$, with corresponding mgf $M(t)$ defined for $|t| \le h_1 < h$ such that $\lim_{n\to\infty} M(t;n) = M(t)$, then Y_n has a limiting distribution with distribution function $F(y)$.

Note that, for $\lim_{n\to\infty} \psi(n)=0$ we have:

$$\lim_{n\to\infty}\left[1+\frac{b}{n}+\frac{\psi(n)}{n}\right]^{cn}=\lim_{n\to\infty}\left[1+\frac{b}{n}\right]^{cn}=e^{bc}$$

Example: Let $Y_n \sim Binomial(n;p)$. Find the limiting distribution of Y_n.

Answer:

$$M(t;n)=E(e^{tY_n})=\left[p.e^t+(1-p)\right]^n \text{ (see pages 22,23)}$$

$$M(t;n)=\left[1+p.e^t-p\right]^n=\left[1+p(e^t-1)\right]^n=\left[1+\frac{np(e^t-1)}{n}\right]^n$$

But $\mu=E(Y_n)=n.p$. Then,

$$M(t;n)=\left[1+\frac{\mu.(e^t-1)}{n}\right]^n$$

$$\lim_{n\to\infty} M(t;n)=e^{\mu(e^t-1)}$$

But this is the mgf of a Poisson random variable with mean μ (see page 23) Then, limiting distribution of Binomial Distribution is Poisson Distribution.

So, for large n (and necessarily small p) Binomial probabilities can be approximated by Poisson distribution.

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5.11. Let X_n have a gamma distribution with parameter $\alpha = n$ and β, where β is not a function of n. Let $Y_n = X_n/n$. Find the limiting distribution of Y_n.

5.11) $X_n \sim Gamma(\alpha = n; \beta)$

$m(t;n) = (1-\beta t)^{-n}; \ t < \frac{1}{\beta}$ → See page 26

$M_{Y_n}(t;n) = E(e^{tY_n}) = E(e^{\frac{t}{n} \cdot X_n}) = m(\frac{t}{n}; n) = (1 - \frac{\beta t}{n})^{-n}; \ \frac{t}{n} < \beta$

$\lim_{n\to\infty} M_{Y_n}(t;n) = \lim_{n\to\infty} (1 - \frac{\beta t}{n})^{-n} = e^{\beta t}$

But this is mgf of a distribution which is degenerate at β because if $f(x) = \begin{cases} 1 & x=\beta \\ 0 & o.w \end{cases}$ then

$M_X(t) = E(e^{tX}) = f(\beta) . e^{\beta t} = e^{\beta t}$

So, $Y_n \xrightarrow{P} \beta$

Additional Properties of Gamma distribution

(i) Remember, $X \sim Exponential(\lambda)$

$f(x) = \lambda . e^{-\lambda x}$ (or $f(x) = \frac{1}{\lambda} \cdot e^{-x/\lambda}$)

Then; $Exponential(\lambda) \equiv Gamma(\alpha = 1; \beta = \lambda)$

(ii) $X \sim Gamma(\alpha, \beta) \Rightarrow \frac{X}{k} \sim Gamma(\alpha; \frac{\beta}{k})$

(iii) $X_i \overset{i.i.d}{\sim} Gamma(\alpha_i; \beta) \Rightarrow \sum_{i=1}^{n} X_i \sim Gamma(\sum_{i=1}^{n} \alpha_i; \beta)$

Note that, (ii) and (iii) can be proved easily using mgf.

5.19. Let $\bar{X}_n$ denote the mean of a random sample of size n from a distribution that has p.d.f. $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere.

(a) Show that the m.g.f. $M(t; n)$ of $Y_n = \sqrt{n}(\bar{X}_n - 1)$ is equal to $[e^{t/\sqrt{n}} - (t/\sqrt{n})e^{t/\sqrt{n}}]^{-n}$, $t < \sqrt{n}$.

(b) Find the limiting distribution of Y_n as $n \to \infty$.

5.19) a) $X_n \sim \text{Exponential}(1) \equiv \text{Gamma}(1, 1)$

$$\bar{X}_n = \frac{\sum X_i}{n}; \quad \sum X_i \sim \text{Gamma}(n, 1) \text{ and}$$

$$\bar{X}_n \sim \text{Gamma}\left(n, \frac{1}{n}\right)$$

Then, $M(t;n) = \left(1 - \frac{t}{n}\right)^{-n}; \ t < n$

$$Y_n = \sqrt{n}(\bar{X}_n - 1) = \sqrt{n}\bar{X}_n - \sqrt{n}$$

By Property (i) of mgf

$$M_{Y_n}(t;n) = e^{-\sqrt{n}t} \cdot M(\sqrt{n}t; n) = e^{-\sqrt{n}t} \cdot \left(1 - \frac{\sqrt{n}t}{n}\right)^{-n}$$

$$M_{Y_n}(t;n) = \left[e^{t/\sqrt{n}} - e^{t/\sqrt{n}} \cdot \frac{t}{\sqrt{n}}\right]^{-n}; \ t < \sqrt{n}$$

b) $$e^{t/\sqrt{n}} = 1 + \frac{t}{\sqrt{n}} + \frac{1}{2!}\cdot\left(\frac{t}{\sqrt{n}}\right)^2 + \underbrace{\frac{1}{3!}\cdot\left(\frac{t}{\sqrt{n}}\right)^3 + \cdots}_{\frac{1}{n}\psi_1(n)}$$ $\quad \sum \frac{x^k}{k!} = e^x$

$$\frac{t}{\sqrt{n}} \cdot e^{t/\sqrt{n}} = \frac{t}{\sqrt{n}} + \frac{t^2}{n} + \underbrace{\frac{1}{2}\frac{t^3}{n\sqrt{n}} + \cdots}_{\frac{1}{n}\psi_2(n)}$$

$$e^{t/\sqrt{n}} - \frac{t}{\sqrt{n}} \cdot e^{t/\sqrt{n}} = 1 - \frac{1}{2}\frac{t^2}{n} + \underbrace{\frac{1}{n}\psi_1(n) - \frac{1}{n}\psi_2(n)}_{\frac{1}{n}\psi(n)} = 1 - \frac{1}{2}t^2/n + \frac{\psi(n)}{n}$$

So; $M_{Y_n}(t;n) = \left(1 - \frac{t^2/2}{n} + \frac{\psi(n)}{n}\right)^{-n}; \ t < \sqrt{n}$

$$\lim_{n\to\infty} M_{Y_n}(t;n) = \lim_{n\to\infty}\left(1 - \frac{t^2}{2} + \frac{\psi(n)}{n}\right)^{-n} = e^{t^2/2}$$

Note that $\lim_{n\to\infty} \frac{\psi(n)}{n} = 0$

Then, limiting distribution of Y_n is standard Normal Distribution

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

Central Limit Theorem

Let $X_1, X_2, \dots, X_n$ denote the observations of a random sample from a distribution with mean μ and variance σ^2. Then the random variable

$$Y_n = \frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X}_n - \mu}{\sigma_{\bar{X}_n}} = \frac{\sqrt{n}.(\bar{X}_n - \mu)}{\sigma}$$

has standard normal limiting distribution.

Note that; $X \sim Normal(\mu; \sigma^2)$

$$m(t) = \exp\left(\frac{\sigma^2 t^2}{2} + \mu t\right)$$

so; $Z \sim Normal(\mu = 0; \sigma^2 = 1^2) \Rightarrow M_z(t) = e^{t^2/2}$

Example: let $Z_i \sim N(0;1)$ and let $Z_1, Z_2, \dots, Z_n$ is a random sample. Find the limiting distribution of $T_n = \frac{\sum z_i + 1}{\sqrt{n}}$

mgf of $N(0;1)$

Answer: $T_n = \frac{\sum z_i + 1}{\sqrt{n}} = \frac{\sum z_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}$; $m(t) = e^{t^2/2}$

$$m_{T_n}(t;n) = E(e^{t.T_n}) = E\left(e^{t.\left(\frac{\sum z_i}{\sqrt{n}} + \frac{1}{\sqrt{n}}\right)}\right) = E\left(e^{t/\sqrt{n}}.e^{t.\frac{\sum z_i}{\sqrt{n}}}\right)$$

$$E(e^{t\sum z_i}) = E(e^{z_1 t}.e^{z_2 t}.\dots.e^{z_n t}) = [m(t)]^n = e^{nt^2/2}$$

$$E(e^{\frac{t}{\sqrt{n}}\sum z_i}) = e^{t^2/2} \text{ and } m_{T_n}(t;n) = e^{t/\sqrt{n}}.e^{t^2/2}$$

$$\lim_{n\to\infty} m_{T_n}(t;n) = e^{t^2/2}$$

Example: Let $X_n \sim \text{Gamma}(1;n)$ and also let $Z_n = \frac{X_n - n}{\sqrt{n}}$. Show that Z_n has standard normal limiting distribution. (Hint: $\ln(1-\delta) = -\delta - (1+\epsilon)\frac{\delta^2}{2}$ when $\delta \to 0; \epsilon \to 0$)

Answer: $m(t;n) = (1-t)^{-n}$

$$M_{Z_n}(t;n) = E(e^{tZ_n}) = E\left(e^{\frac{t}{\sqrt{n}}X_n} \cdot e^{-\frac{n}{\sqrt{n}}\cdot t}\right) = e^{-\sqrt{n}t} \cdot m\left(\tfrac{t}{\sqrt{n}};n\right)$$

$$= e^{-\sqrt{n}t} \cdot \left(1 - \frac{t}{\sqrt{n}}\right)^{-n} = e^{-\sqrt{n}t} \cdot e^{-n\ln\left(1-\frac{t}{\sqrt{n}}\right)}$$

$$= \exp\left\{-\sqrt{n}t - n\ln\left(1 - \underbrace{\frac{t}{\sqrt{n}}}_{=\delta}\right)\right\} = \exp\left\{-\sqrt{n}t - n\left(-\frac{t}{\sqrt{n}} - \underbrace{(1+\epsilon)\frac{t^2}{2n}}_{\delta^2/2}\right)\right\}$$

$$= \exp\left\{-\sqrt{n}t + \sqrt{n}t + (1+\epsilon)\frac{t^2}{2}\right\} = \exp\left\{(1+\epsilon)\cdot\frac{t^2}{2}\right\}$$

$$M_{Z_n}(t;n) = \exp\left\{(1+\epsilon)\frac{t^2}{2}\right\}$$

$$\lim_{n\to\infty} M_{Z_n}(t;n) = \lim_{\substack{n\to\infty \\ \epsilon \to 0}} \exp\left\{(1+\epsilon)\frac{t^2}{2}\right\} = e^{t^2/2}$$

5.20. Let $\bar{X}$ denote the mean of a random sample of size 100 from a distribution that is $\chi^2(50)$. Compute an approximate value of $\Pr(49 < \bar{X} < 51)$.

5.21. Let $\bar{X}$ denote the mean of a random sample of size 128 from a gamma distribution with $\alpha = 2$ and $\beta = 4$. Approximate $\Pr(7 < \bar{X} < 9)$.

5.22. Let Y be $b(72, \frac{1}{3})$. Approximate $\Pr(22 \le Y \le 28)$.

5.23. Compute an approximate probability that the mean of a random sample of size 15 from a distribution having p.d.f. $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere, is between $\frac{3}{5}$ and $\frac{4}{5}$.

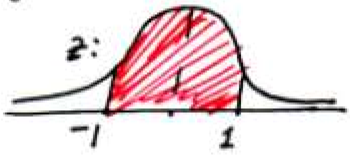
5.20) $X \sim \chi^2_{(50)} \quad \exists\, \mu = E(X) = 50 \; ; \; \sigma^2 = Var(X) = 2 \cdot 50 = 100$

$\sigma = \sqrt{100} = 10$

$n = 100$

$$P(49 < \bar{X} < 51) = P\Big(\frac{49-50}{10/\sqrt{100}} < \underbrace{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}_{Z \sim N(0,1)} < \frac{51-50}{\sigma/\sqrt{n}}\Big)$$

$$= P(-1 < Z < 1) = 2 \cdot 0{,}3413 = 0{,}6826$$



5.21) $X \sim Gamma\,(\alpha = 2;\ \beta = 4)$

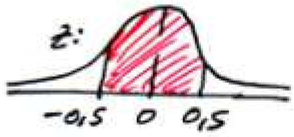
$\mu = E(X) = \alpha\beta = 2 \cdot 4 = 8 \; ; \; \sigma^2 = Var(X) = \alpha\beta^2 = 2 \cdot 4^2 = 32$

$\sigma = \sqrt{32} = 4\sqrt{2}$

$n = 128$

$$P(7 < \bar{X} < 9) = P\Big(\frac{7-8}{4\sqrt{2}/\sqrt{128}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{9-8}{4\sqrt{2}/\sqrt{128}}\Big) = P(-0{,}5 < Z < 0{,}5)$$

$$= 2 \cdot 0{,}1915 = 0{,}383$$



5.22) $Y \sim Binomial\,(n = 72;\ p = \frac{1}{3})$

$\mu = np = 72 \cdot \frac{1}{3} = 24 \; ; \; \sigma^2 = n.p.(1-p) = 72 \cdot \frac{1}{3} \cdot \frac{2}{3} = 16$

$\sigma = \sqrt{16} = 4$

continuity correction

$$P(22 < Y < 28) = P(23 \le Y \le 27) = P\Big(\frac{22{,}5-24}{4} \le \frac{Y-\mu}{\sigma} \le \frac{27{,}5-24}{4}\Big)$$

$$= P(-0{,}38 < Z < 0{,}88) = 0{,}1480 + 0{,}3116 = 0{,}4596$$

5.23)

$$f(x) = 3x^2 \quad 0 < x < 1$$
$$= 0 \quad o.w$$

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 x \cdot 3x^2\,dx = 3 \cdot \frac{x^4}{4}\Big]_0^1 = \frac{3}{4} - 0 = \frac{3}{4} = 0{,}75$$

$$E(X^2) = \int_0^1 x^2 \cdot 3x^2\,dx = 3 \cdot \frac{x^5}{5}\Big]_0^1 = \frac{3}{5}$$

$$\sigma^2 = Var(X) = E(X^2) - \mu^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0{,}0375$$

$$\sigma = \sqrt{0{,}0375} = 0{,}194 \text{ and } n = 15$$

$$P\left(\frac{3}{5} < \bar{X} < \frac{4}{5}\right) = P(0{,}6 < \bar{X} < 0{,}8) = P\left(\frac{0{,}6 - 0{,}75}{0{,}194/\sqrt{15}} < \underbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}_{= z} < \frac{0{,}8 - 0{,}75}{0{,}194/\sqrt{15}}\right)$$

$$= (-3 < z < 1) = 0{,}4986 + 0{,}3413 = 0{,}8399$$

z:

Some Theorems on Limiting Distributions

THM(I) Let $F_n(u)$ is the distribution function of U_n. Also let $U_n \xrightarrow[p]{} c$ where c is a constant.

Then; $\frac{U_n}{c} \xrightarrow[p]{} 1$

THM(II) Let $U_n \xrightarrow[p]{} c$ and $P(U_n < 0) = 0$ for every n.

Then $\sqrt{U_n} \rightarrow \sqrt{c}$

THM(III) Let U_n has distribution function $F_n(u)$ and has limiting distribution $F(u)$.

Also let $V_n \xrightarrow[p]{} 1$

Then, $W_n = \frac{U_n}{V_n}$ has limiting distribution $F(w)$

THM(IV) Let $U_n \xrightarrow[p]{} U$ and $V_n \xrightarrow[p]{} V$. Then;

(i) $U_n + V_n \xrightarrow[p]{} U + V$

(ii) $U_n \cdot V_n \xrightarrow[p]{} U \cdot V$

(iii) $aU_n + bV_n \xrightarrow[p]{} aU + bV$

(iv) $\frac{U_n}{V_n} \xrightarrow[p]{} \frac{U}{V}$ provided $V_n \neq 0, V \neq 0$

Example: Show that $\frac{Y_n - np}{\sqrt{n \cdot \left(\frac{Y_n}{n}\right)\left(1 - \frac{Y_n}{n}\right)}}$ converges in distribution to $N(0;1)$ where $Y_n \sim Binomial(n;p)$

Answer: $E(Y_n) = n.p$ and $Var(Y_n) = n.p.(1-p)$. So, by CLT;

$$\Rightarrow U_n = \frac{Y_n - np}{\sqrt{np(1-p)}} \underset{appr.}{\sim} Normal(0;1)$$

Moreover; $\frac{y_n}{n} \xrightarrow[p]{} p$ and $1-\frac{y_n}{n} \xrightarrow[p]{} 1-p$.

So, by Thm IV-(ii), $\frac{y_n}{n}\left(1-\frac{y_n}{n}\right) \xrightarrow[p]{} p(1-p)$

And by Thm I, $$\frac{\frac{y_n}{n}\left(1-\frac{y_n}{n}\right)}{p(1-p)} \xrightarrow[p]{} 1$$

By Thm II, $$V_n = \sqrt{\frac{\frac{y_n}{n}\left(1-\frac{y_n}{n}\right)}{p(1-p)}} \xrightarrow[p]{} \sqrt{1} = 1$$

If we let $W_n = \frac{U_n}{V_n} = \frac{y_n - np}{\sqrt{n\left(\frac{y_n}{n}\right)\left(1-\frac{y_n}{n}\right)}}$ which is the expression we want to show, by Thm III, W_n has same limiting dist. with U_n.

Then; $$\frac{y_n - np}{n\left(\frac{y_n}{n}\right)\left(1-\frac{y_n}{n}\right)} \underset{\text{app.}}{\sim} \text{Normal}(0;1)$$

Example: let $\bar{X}_n$ & S_n^2 be mean and variance of a random sample of size n from Normal$(\mu;\sigma^2)$.
Let $W_n = \frac{\bar{X}_n}{S_n/\sigma}$. Show that $W_n \xrightarrow[p]{} \mu$

Answer: We know that $\bar{X}_n \xrightarrow[p]{} \mu$ and $S_n^2 \xrightarrow[p]{} \sigma^2$

By Thm. I, $\frac{S_n^2}{\sigma^2} \xrightarrow[p]{} 1$

and By Thm II, $\sqrt{\frac{S_n^2}{\sigma^2}} = \frac{S_n}{\sigma} \xrightarrow[p]{} \sqrt{1} = 1$

By Thm III, W_n has same limiting distribution with $\bar{X}_n$. So; $W_n \xrightarrow[p]{} \mu$.

5.33. Let $\bar{X}_n$ denote the mean of a random sample of size n from a gamma distribution with parameters $\alpha = \mu > 0$ and $\beta = 1$. Show that the limiting distribution of $\sqrt{n}(\bar{X}_n - \mu)/\sqrt{\bar{X}_n}$ is $N(0, 1)$.

5.34. Let $T_n = (\bar{X}_n - \mu)/\sqrt{S_n^2/(n-1)}$, where $\bar{X}_n$ and S_n^2 represent, respectively, the mean and the variance of a random sample of size n from a distribution that is $N(\mu, \sigma^2)$. Prove that the limiting distribution of T_n is $N(0, 1)$.

5.33) $X_i \sim Gamma(\alpha = \mu; \beta = 1)$

$E(X_i) = \alpha.\beta = \mu$ and $Var(X_i) = \alpha.\beta^2 = \mu$

$E(\bar{X}_n) = \mu$ and $Var(\bar{X}_n) = \frac{\mu}{n}$

By C.L.T, $U_n = \frac{\bar{X}_n - \mu}{\sqrt{\mu/n}} \underset{appr.}{\sim} Normal(0; 1)$

By weak law of large Numbers; $\bar{X}_n \xrightarrow[p]{} E(\bar{X}_n) = \mu$

By Thm II, $\sqrt{\bar{X}_n} \xrightarrow[p]{} \sqrt{\mu}$ and By Thm I, $V_n = \sqrt{\frac{\bar{X}_n}{\mu}} \longrightarrow 1$

Finally, By Thm III, $W_n = \frac{U_n}{V_n} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sqrt{\bar{X}_n}} \underset{appr.}{\sim} Normal(0; 1)$

5.34) By CLT, $U_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim Normal(0; 1)$

We have shown that $\frac{S_n}{\sigma} \xrightarrow[p]{} 1$

But since $\frac{n}{n-1} \xrightarrow[n \to \infty]{} 1$; $V_n = \frac{n . S_n}{(n-1)\sigma} \xrightarrow[p]{} 1$ is also true.

Then, $W_n = \frac{U_n}{V_n} = \frac{(\bar{X}_n - \mu)}{\sqrt{S_n^2/n-1}} \sim Normal(0; 1)$ by Thm III