OPTIMIZATION Lecture Notes | CHAPTER 3-I

What is a Linear Programming Problem?

An LP problem is an optimization problem in which the goal is either maximize (usually profit) or minimize (usually cost) an objective function

* "Decision Variables" are the variables of the objective function and each have a unit profit (or cost). Namely, each variable x_i has a unit value c_i. Then, the general form of the objective function is

$$\boxed{Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n}$$

* Decision Variables can NOT take values freely. That's why we have an optimization problem. In General, in a maximization problem, we have limited available sources that constraints the objective function and in a minimization problem, we have requirements to be satisfied that constraints the objective function.

Each constraint is a linear equality or inequality of the following form

$$\boxed{a_{j1} x_1 + a_{j2} x_2 + \cdots + a_{jn} x_n \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} b_j}$$

2 Farmer Jones bakes two types of cake (chocolate and vanilla) to supplement his income. Each chocolate cake can be sold for \$1, and each vanilla cake can be sold for 50¢. Each chocolate cake requires 20 minutes of baking time and uses 4 eggs. Each vanilla cake requires 40 minutes of baking time and uses 1 egg. Eight hours of baking time and 30 eggs are available. Formulate an ~~LP~~ to maximize Farmer Jones's revenue

2) what to produce?

x_1: # of chocolate cake baked

x_2: # of vanilia cakes baked

These have unit profits!

	TIME	EGG	PROFIT
(x_1) Chocolate cake	20min.	4 eggs	100¢
(x_2) Vanilia cake	40min.	1 egg	50¢
AVAILABLE	8.60min	30 eggs	

Maximize $z = 100x_1 + 50x_2$

subject to

$20x_1 + 40x_2 \leq 480$ → Time Constraint

$4x_1 + x_2 \leq 30$ → Egg Constraint

$x_1, x_2 \geq 0$ nonneg.

1 Farmer Jones must determine how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. All wheat can be sold at \$4 a bushel, and all corn can be sold at \$3 a bushel. Seven acres of land and 40 hours per week of labor are available. Government regulations require that at least 30 bushels of corn be produced during the current year. Let x_1 = number of acres of corn planted, and x_2 = number of acres of wheat planted. Using these decision variables, formulate an LP whose solution will tell Farmer Jones how to maximize the total revenue from wheat and corn.

1) what to decide?

x_1: # of acres of corn planted

x_2: # of acres of wheat planted.

An acre of wheat: 25 bushels of wheat
req. 10 hours of labor

An acre of corn: 10 bushels of corn
req. 4 hours of labor.

Revenue: wheat ⇒ 4\$/bushel
corn ⇒ 3\$/bushel

Available land: 7 acre

Available Labor: 40 hours

Goverment reg: At least 30 bushels of corn.

$$\max.\ z = \underbrace{30}_{=3.10}x_1 + \underbrace{100}_{4.25}x_2$$

subject to

$x_1 + x_2 \leq 7$ → Acre Const.

$4x_1 + 10x_2 \leq 40$ → Labor Const.

$10x_1 \geq 30$ → Gov. Reg. Const.

$x_1, x_2 \geq 0$ nonnegativity

Feasible Region & Optimal Solution

* The feasible region for an LP is the set of all points that satisfies all the LP's constraints and sign restrictions

* For a max. problem, an optimal solution to an LP is a point in the feasible region with the largest objective function value. Similarly, for a min. problem, an optimal solution is a point in the feasible region with the smallest obj. function value.

2 Answer these questions about Problem 1.
a Is ($x_1 = 2, x_2 = 3$) in the feasible region?
b Is ($x_1 = 4, x_2 = 3$) in the feasible region?
c Is ($x_1 = 2, x_2 = -1$) in the feasible region?
d Is ($x_1 = 3, x_2 = 2$) in the feasible region?

2) a) x_1 x_2 rhs

$2 + 3 \leq 7$ → acre ✓

$4 \cdot 2 + 10 \cdot 3 \leq 40$ → labor ✓

$10 \cdot 3 \geq 30$ → Gov. ✓ YES!

$x_1, x_2 \geq 0$ → Nonnegativeety ✓

b) NO: $4 \cdot 4 + 10 \cdot 3 = 46 \not\leq 40$
Labor Constraint

c) NO: $x_2 = -1 \not\geq 0$
Nonnegativeety Constraint

d) $10 \cdot 2 \not\geq 30$
Goverment Regulations Constraint

The Graphical Solution of Two Variable LP Problems:

(i) Draw each constraint's equations in (x_1, x_2) plane

(ii) Find the feasible region. This is the region that all constraint's shaded areas intersect

(iii) Find the values of objective function at each corner point of the feasible region

(iv) Determine the optimum value of z and values of x_1 and x_2 as the solution.

Alp Giray Özen | 0533 549 91 08 | alp@lecturemania.com | www.lecturemania.com

4 Truckco manufactures two types of trucks: 1 and 2. Each truck must go through the painting shop and assembly shop. If the painting shop were completely devoted to painting Type 1 trucks, then 800 per day could be painted; if the painting shop were completely devoted to painting Type 2 trucks, then 700 per day could be painted. If the assembly shop were completely devoted to assembling truck 1 engines, then 1,500 per day could be assembled; if the assembly shop were completely devoted to assembling truck 2 engines, then 1,200 per day could be assembled. Each Type 1 truck contributes \$300 to profit; each Type 2 truck contributes \$500. Formulate an LP that will maximize Truckco's profit.

4)

	Painting	Assembly	UNIT PROFIT
(x_1) Truck 1	800/day	1500/day	300\$
(x_2) Truck 2	700/day	1200/day	500\$

⇓

	Painting	Assembly	PROFIT
(x_1) Truck 1	$\frac{1}{800}$ days	$\frac{1}{1500}$ days	300\$
(x_2) Truck 2	$\frac{1}{700}$ days	$\frac{1}{1200}$ days	500\$
AVAILABLE	1	1	

Maximize

$$z = 300x_1 + 500x_2$$

s.t. (i) $\frac{x_1}{800} + \frac{x_2}{700} \le 1$ → Painting constraint

(ii) $\frac{x_1}{1500} + \frac{x_2}{1200} \le 1$ → Assembly Constraint.

$x_1, x_2 \ge 0$

(i) $\frac{x_1}{800} + \frac{x_2}{700} = 1$

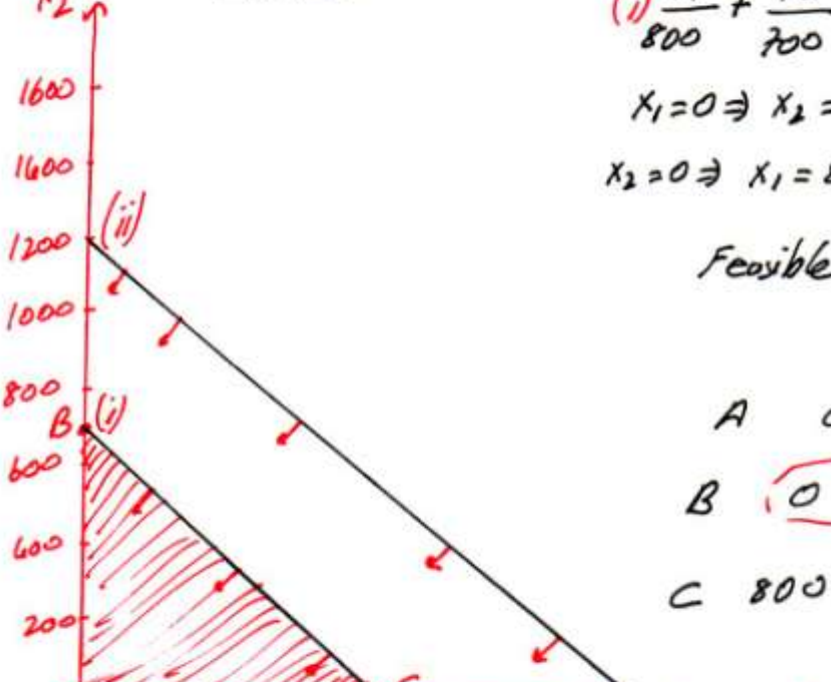
$x_1 = 0 \Rightarrow x_2 = 700$

$x_2 = 0 \Rightarrow x_1 = 800$

(ii) $\frac{x_1}{1500} + \frac{x_2}{1200} = 1$

$x_1 = 0 \Rightarrow x_2 = 1200$

$x_2 = 0 \Rightarrow x_1 = 1500$



Feasible Set (or Corner Points): A, B, C

	x_1	x_2	$z = 300x_1 + 500x_2$
A	0	0	0
B	0	700	$z = 500 \cdot 700 = \boxed{350000}$ max
C	800	0	$z = 300 \cdot 800 = 240000$

Maximum profit is obtained when NO Type-I trucks and 700 Type-II trucks are produced. Maximum profit is 350 thousand \$

Note that, constraint (ii) is a redundant constraint.

3 Leary Chemical manufactures three chemicals: A, B, and C. These chemicals are produced via two production processes: 1 and 2. Running process 1 for an hour costs \$4 and yields 3 units of A, 1 of B, and 1 of C. Running process 2 for an hour costs \$1 and produces 1 unit of A and 1 of B. To meet customer demands, at least 10 units of A, 5 of B, and 3 of C must be produced daily. Graphically determine a daily production plan that minimizes the cost of meeting Leary Chemical's daily demands.

3) x_1: Process 1 running time
x_2: Process 2 running time
because they have costs.

	A	B	C	costs
(x_1) Proc. 1	3	1	1	4\$
(x_2) Proc. 2	1	1	0	1\$
Required	10	5	3	

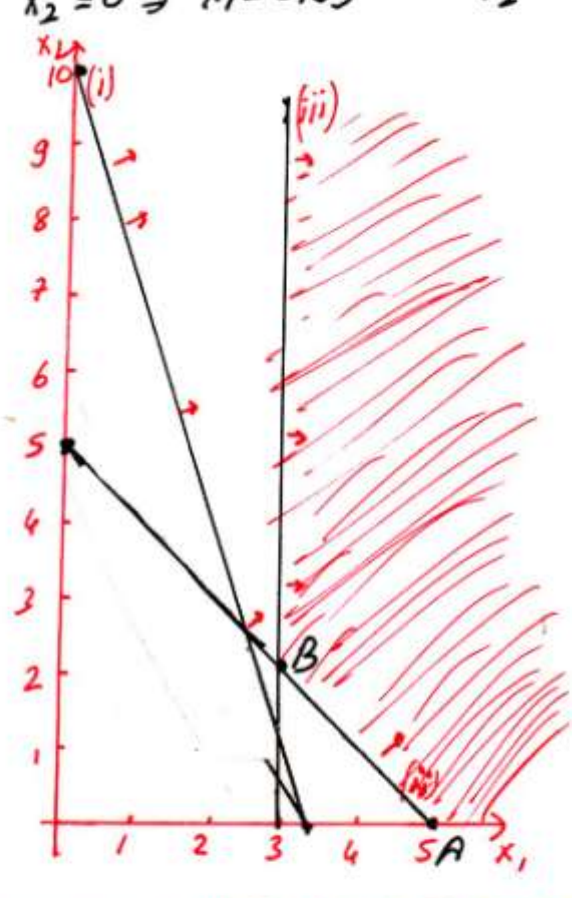
Minimize $z = 4x_1 + x_2$

subject to (i) $3x_1 + x_2 \geq 10$ → Chem. A constraint
(ii) $x_1 + x_2 \geq 5$ → Chem. B Constraint
(iii) $x_1 \geq 3$ → Chem. C constraint

(i) $3x_1 + x_2 = 10$
$x_1 = 0 \Rightarrow x_2 = 10$
$x_2 = 0 \Rightarrow x_1 = 3{,}33$

(ii) $x_1 + x_2 = 5$
$x_1 = 0 \Rightarrow x_2 = 5$
$x_2 = 0 \Rightarrow x_1 = 5$

(iii) $x_1 = 3$



Feasible set: {A, B}

B: (ii) & (iii)
$x_1 + x_2 = 5$
$x_1 = 3$
$x_1 = 3 \quad x_2 = 2$

	x_1	x_2	$z = 4x_1 + x_2$
A	5	0	$z = 4.5 = 20$
B	3	2	$z = 4.3 + 2 = \boxed{14}$ ← min

Minimum Cost is obtained when Proc. 1 is used for 3 hours and Proc. 2 is used for 2 Hours and Min. cost is 14 \$.
Note that, Const. (i) is redundant.

Isoprofit Line & Binding Constraints

Consider the case of two-variable LP problem. If we give a constant value z_0 to the objective function $z = c_1x_1 + c_2x_2$, we obtain a line in (x_1, x_2) plane on which we obtain the value of z_0. The line

$$\boxed{c_1x_1 + c_2x_2 = z_0}$$

is isoprofit line for a max. problem and isocost line for a min. problem.

Consider a usual max. problem with two variables and let the isoprofit line is NOT parallel any of the constraint lines. If the isoprofit line has a line segment in the feasible region, it means that we can increase z_0.

Note that, the vector $\vec{v} = (c_1, c_2)$ is perpendicular to the isoprofit line $c_1x_1 + c_2x_2 = z_0$ and the profit increases in the direction of $\vec{v}$ until the line leaves the feasible region. Since we have assumed that isoprofit line is NOT parallel to any of the constraints, the corner point (x_1^*, x_2^*) will yield $c_1x_1^* + c_2x_2^* = z_{max}$

The constraints that intersects at this point $(x_1^*; x_2^*)$ are called binding constraints. It is clear that, the sources are used up at this binding constraints because we have the equalities for these constraints.

The following example illustrates these ideas.

4. The Pinewood Furniture Company produces chairs and tables from two resources—labor and wood. The company has 80 hours of labor and 36 pounds of wood available each day. Demand for chairs is limited to 6 per day. Each chair requires 8 hours of labor and 2 pounds of wood, whereas a table requires 10 hours of labor and 6 pounds of wood. The profit derived from each chair is $400 and from each table, $100. The company wants to determine the number of chairs and tables to produce each day in order to maximize profit.
   a. Formulate a linear programming model for this problem.
   b. Solve this model by using graphical analysis.

5. In Problem 4, how much labor and wood will be unused if the optimal numbers of chairs and tables are produced?

6. In Problem 4, explain the effect on the optimal solution of changing the profit on a table from $100 to $500.

4) a)

	Labor	Wood	PROFIT
(x_1) Chairs	8	2	400
(x_2) Tables	10	6	100
AVAILABLE	80	36	

Maximize $z = 400x_1 + 100x_2$

subject to (i) $8x_1 + 10x_2 \leq 80$ → Labor Constraint

(ii) $2x_1 + 6x_2 \leq 36$ → Wood Constraint

(iii) $x_1 \leq 6$ → Demand Constraint.

$x_1, x_2 \geq 0$

b) (i) $8x_1 + 10x_2 = 80$ (ii) $2x_1 + 6x_2 = 36$ (iii) $x_1 = 6$

$x_1 = 0 \Rightarrow x_2 = 8$ $x_1 = 0 \Rightarrow x_2 = 6$

$x_2 = 0 \Rightarrow x_1 = 10$ $x_2 = 0 \Rightarrow x_1 = 18$

(i) & (ii)

$8x_1 + 10x_2 = 80$

-4 $2x_1 + 6x_2 = 36$

$8x_1 + 10x_2 = 80$

$-8x_1 - 24x_2 = -144$

$-14x_2 = -64$

$\boxed{x_2 = 4,57}$

$2x_1 + 6 \cdot 4,57 = 36$

$2x_1 = 8,57$

$\boxed{x_1 = 4,28}$

(i) & (iii)

$\boxed{x_1 = 6}$

$8x_1 + 10x_2 = 80$

$8 \cdot 6 + 10x_2 = 80$

$10x_2 = 32$

$\boxed{x_2 = 3,2}$

(ii) & (iii)

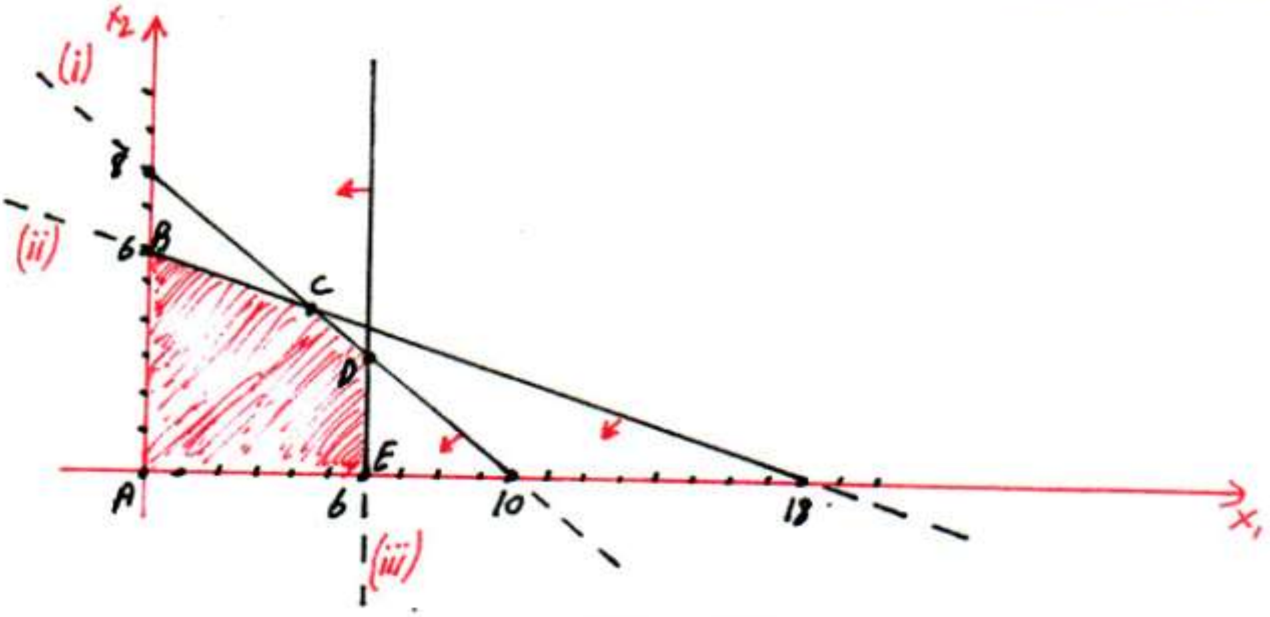
$\boxed{x_1 = 6}$

$2x_1 + 6x_2 = 36$

$2 \cdot 6 + 6x_2 = 36$

$6x_2 = 24$

$\boxed{x_2 = 4}$



$z = 400x_1 + 100x_2$

	x_1	x_2	
A	0	0	$z = 400 \cdot 0 + 100 \cdot 0 = 0$
B	0	6	$z = 400 \cdot 0 + 100 \cdot 6 = 600$
C	4,28	4,57	$z = 400 \cdot 4{,}28 + 100 \cdot 4{,}57 = 2169$
D	6	3,2	$z = 400 \cdot 6 + 100 \cdot 3{,}2 = \underline{2720}$ → Maximum
E	6	0	$z = 400 \cdot 6 + 100 \cdot 0 = 2400$

Maximum profit is 2720 when 6 chairs and 3,2 tables are produced.

5) Labor Used: $8 \cdot 6 + 10 \cdot 3{,}2 = 80$

Unused Labor $= 80 - 80 = 0$

Wood Used: $2 \cdot 6 + 6 \cdot 3{,}2 = 31{,}2$

Unused Wood $= 36 - 31{,}2 = 4{,}8$

6) $z = 400x_1 + 500x_2$

	x_1	x_2	
A	0	0	$z = 400 \cdot 0 + 500 \cdot 0 = 0$
B	0	6	$z = 400 \cdot 0 + 500 \cdot 6 = 3000$
C	4,28	4,57	$z = 400 \cdot 4{,}28 + 500 \cdot 4{,}57 = 4000$
D	6	3,2	$z = 400 \cdot 6 + 500 \cdot 3{,}2 = \underline{4000}$
E	6	0	$z = 400 \cdot 6 + 500 \cdot 0 = 2400$

One of the Maximums

Maximum profit increases to 4000 but x_1, x_2 values do NOT change.

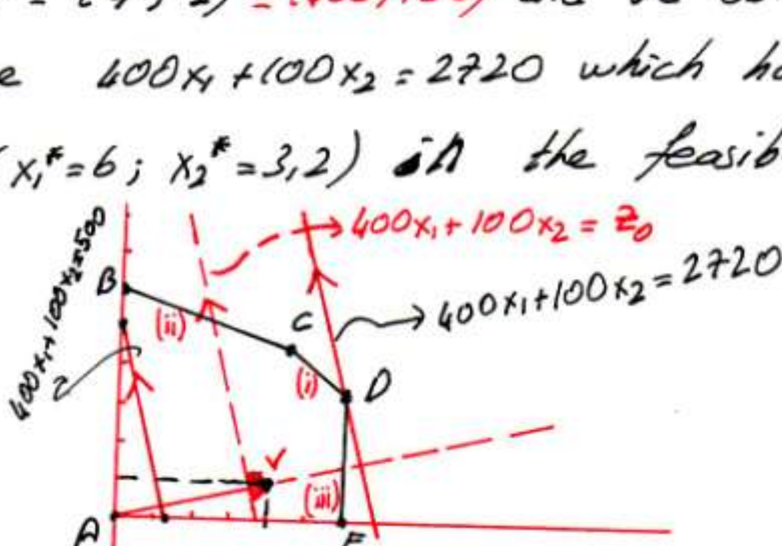
*Note that, the maximum profit $z_{max} = 2720$ occurs at the intersection of the (i) Labor constraint and (iii) Demand Constraint. Then, we say that Constraints (i) and (iii) are binding constraints and constraint (ii) is a non-binding constraint.

More formally, a constraint is binding if the left hand side and the right hand side of the constraints are equal when the optimal values of the decision variables are submitted into the constraint.

Also, that's why (i) Available Labor and (iii) Max. Demand are used up at the point $D(x_1^* = 6;\ x_2^* = 3{,}2)$ and (ii) Available Wood is NOT used up!

*Now, consider the isoprofit line $400x_1 + 100x_2 = 500$

As we noted earlier, the profit increases from $z_0 = 500$ up to $z_{max} = 2720$ in the direction of the vector $v = (4;1) = (400;100)$ and we obtain the last isoprofit line $400x_1 + 100x_2 = 2720$ which has only one point $D(x_1^* = 6;\ x_2^* = 3{,}2)$ in the feasible region.



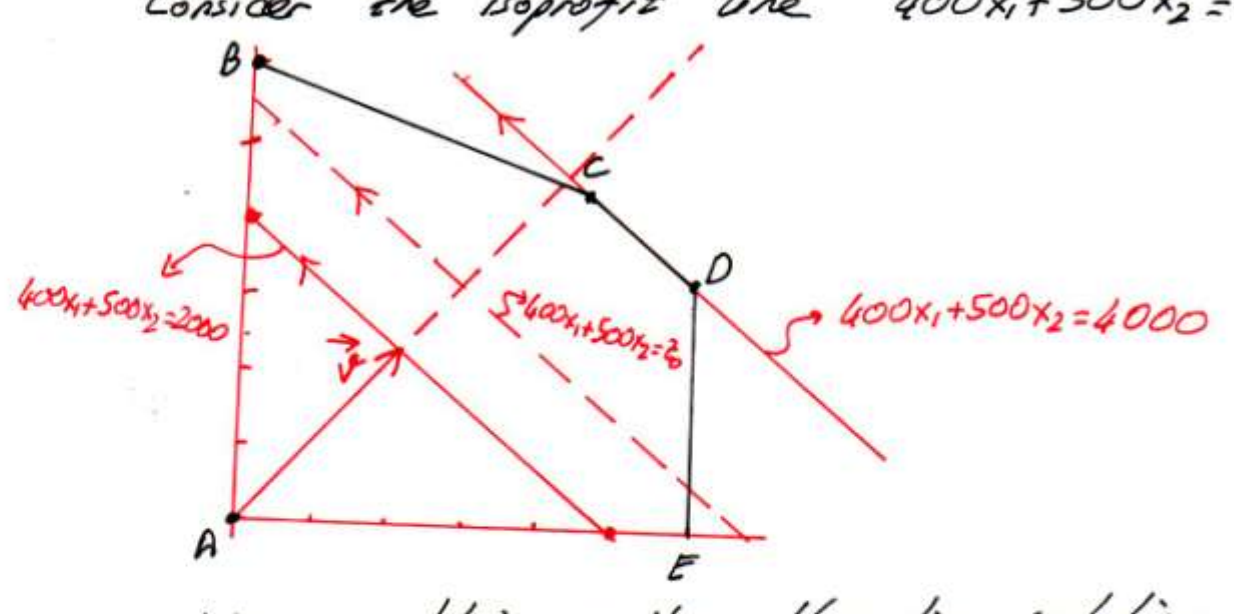
* when we change our objective function to $z = 400x_1 + 500x_2$, we have $z_{max} = 4000$ both at point C and at point D. The reason is that,

(i) $400x_1 + 500x_2 = z_0$ isoprofit line and

(ii) $8x_1 + 10x_2 = 80$ labor constraint equation

are parallel. Then, the isoprofit lines leaves the feasible region at line segment [CD] and all the points at [CD] has profit $z_{max} = 4000$

Consider the isoprofit line $400x_1 + 500x_2 = 2000$



We can obtain another alternative solution P in the line segment [CD] by letting $0 \leq k \leq 1$ and

$$P = k \cdot C + (1-k) \cdot D$$

for example, for $k = 0{,}6$, we have

$$P = 0{,}6 \cdot (4{,}28; 4{,}57) + 0{,}4 \cdot (6; 3{,}2)$$

$$P = (0{,}6 \cdot 4{,}28 + 0{,}4 \cdot 6 ; 0{,}6 \cdot 4{,}57 + 0{,}4 \cdot 3{,}2) = (4{,}97; 4{,}02)$$

And surely, $z(P) = 400 \cdot 4{,}97 + 500 \cdot 4{,}02 = 4\,000 = z_{max}$

Infeasible LP & Unbounded LP

* If an LP's feasible region is empty (contain no points), it is called an infeasible LP. Since the optimal solution to an LP is the best point in the feasible region, an infeasible LP has no optimal solution.

* A max. problem is unbounded if there are points that always stay in the feasible region when we move parallel to our original isoprofit line in the direction of increasing z (that is $\vec{v}$). Likewise, a min. problem is unbounded if we never leave the feasible region when moving in the direction of decreasing z (that is $-\vec{v}$)

As we have seen so far, every LP must fall into one of the following four cases:

(i) The LP has a unique optimal solution

(ii) The LP has alternative or multiple optimal solutions: Two ~~or~~ more extreme points are optimal, and the LP will have infinite # of optimal solutions.

(iii) The LP is infeasible: The feasible region contain NO points.

(iv) The LP is unbounded: There are points in the feasible region with arbitrarily large z-values (max problem) or arbitrarily small z-values (min. problem)

Identify which of Cases 1–4 apply to each of the following LPs:

1

$$\max z = x_1 + x_2$$
s.t. (i) $x_1 + x_2 \le 4$
(ii) $x_1 - x_2 \ge 5$
$x_1, x_2 \ge 0$

2

$$\max z = 4x_1 + x_2$$
s.t. (i) $8x_1 + 2x_2 \le 16$
(ii) $5x_1 + 2x_2 \le 12$
$x_1, x_2 \ge 0$

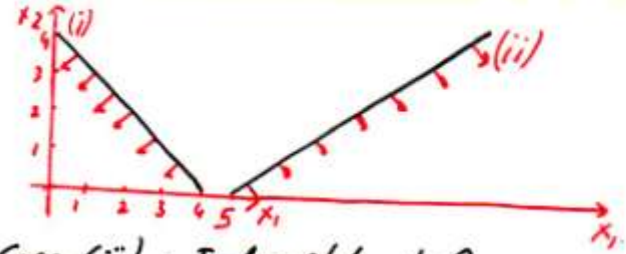
3

$$\max z = -x_1 + 3x_2$$
s.t. (i) $x_1 - x_2 \le 4$
(ii) $x_1 + 2x_2 \ge 4$
$x_1, x_2 \ge 0$

4

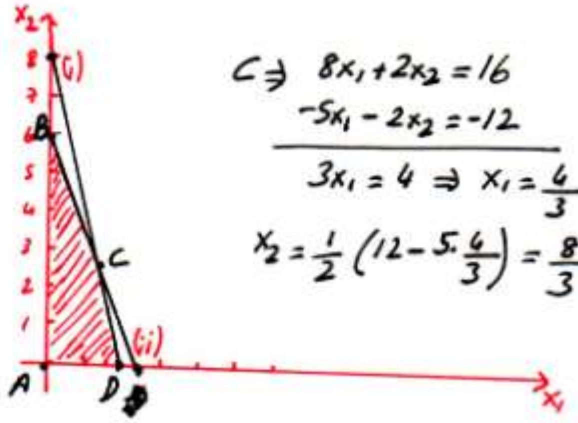
$$\max z = 3x_1 + x_2$$
s.t. (i) $2x_1 + x_2 \le 6$
(ii) $x_1 + 3x_2 \le 9$
$x_1, x_2 \ge 0$

1)



Case (iii) : Infeasible LP

2)



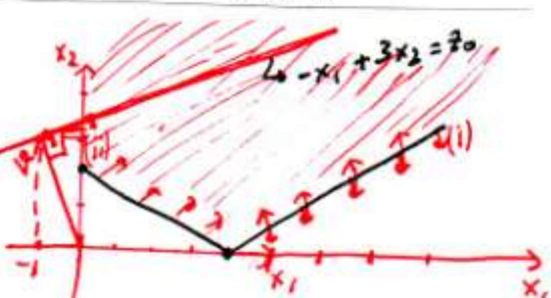
$C \Rightarrow \quad 8x_1 + 2x_2 = 16$
$-5x_1 - 2x_2 = -12$
$3x_1 = 4 \Rightarrow x_1 = \frac{4}{3}$
$x_2 = \frac{1}{2}\left(12 - 5 \cdot \frac{4}{3}\right) = \frac{8}{3}$

$z = 4x_1 + x_2$

$A(0;0) \quad z = 0$
$B(0;6) \quad z = 4$
$C(4/3;8/3) \quad z = 4 \cdot \frac{4}{3} + \frac{8}{3} = \frac{24}{3} = \boxed{8}$
$D(2;0) \quad z = 4 \cdot 2 = \boxed{8} \quad z_{max} = 8$

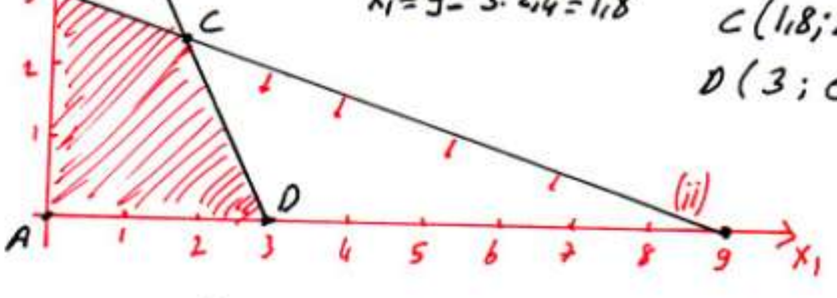
Case (ii) : Alternative Solutions

3)



Izoprofit line can increase unboundadely in the direction: $\vec{a}$
Case (iv): Unbounded LP

4)



$C \Rightarrow \quad 2x_1 + x_2 = 6$
$-2x_1 - 6x_2 = -18$
$-5x_2 = -12$
$x_2 = 2{,}4$
$x_1 = 9 - 3 \cdot 2{,}4 = 1{,}8$

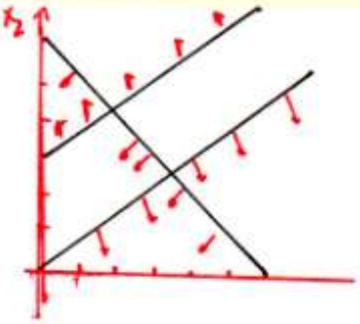
$z = 3x_1 + x_2$

$A(0;0) \quad z = 0$
$B(0;3) \quad z = 3$
$C(1{,}8;2{,}4) \quad z = 3 \cdot 1{,}8 + 2{,}4 = 7{,}8$
$D(3;0) \quad z = 3 \cdot 3 = \boxed{9} \quad z_{max} = 9$

Case (i): Unique Optimal Solution.

8 Graphically find all optimal solutions to the following LP:

$$\min z = x_1 - x_2$$
s.t. (i) $x_1 + x_2 \le 6$
(ii) $x_1 - x_2 \ge 0$
(iii) $x_2 - x_1 \ge 3$
$x_1, x_2 \ge 0$



Case (iii): Infeasible LP

26 Graphically solve the following LP:

$$\min z = 5x_1 + x_2$$
s.t. (i) $2x_1 + x_2 \ge 6$
(ii) $x_1 + x_2 \ge 4$
(iii) $2x_1 + 10x_2 \ge 20$
$x_1, x_2 \ge 0$

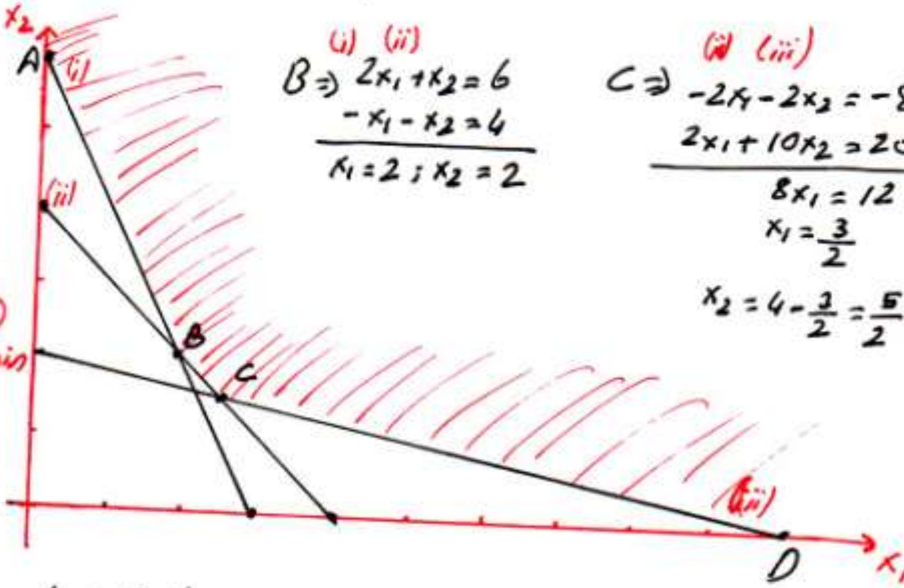
$z = 5x_1 + x_2$

$A(0;6)$ opt. $z = 5.0 + 6 = 6$ z_{min}

$B(2;2)$ $z = 5.2 + 2 = 12$

$C(1,5;2,5)$ $z = 5.1,5 + 2,5 = 10$

$D(10;0)$ $z = 5.10 = 50$



(i) (ii)
$B \Rightarrow 2x_1 + x_2 = 6$
$-x_1 - x_2 = 4$
$x_1 = 2 \,;\, x_2 = 2$

(ii) (iii)
$C \Rightarrow -2x_1 - 2x_2 = -8$
$2x_1 + 10x_2 = 20$
$8x_1 = 12$
$x_1 = \frac{3}{2}$
$x_2 = 4 - \frac{3}{2} = \frac{5}{2}$

Case(i): Unique optimal Solution

31 Graphically find all solutions to the following LP:

$$\min z = x_1 - 2x_2$$
s.t. $x_1 \ge 4$
$x_1 + x_2 \ge 8$
$x_1 - x_2 \le 6$
$x_1, x_2 \ge 0$

The isocost line can decrease unboundadely in the direction of $-\vec{v}$

Case (iv): Unbounded LP!

Be careful!
Corner Points do NOT give an optimum solution.