OPTIMIZATION Lecture Notes | CHAPTER 3-II

MORE MODELING EXAMPLES

* Transportation & Transshipment Problem

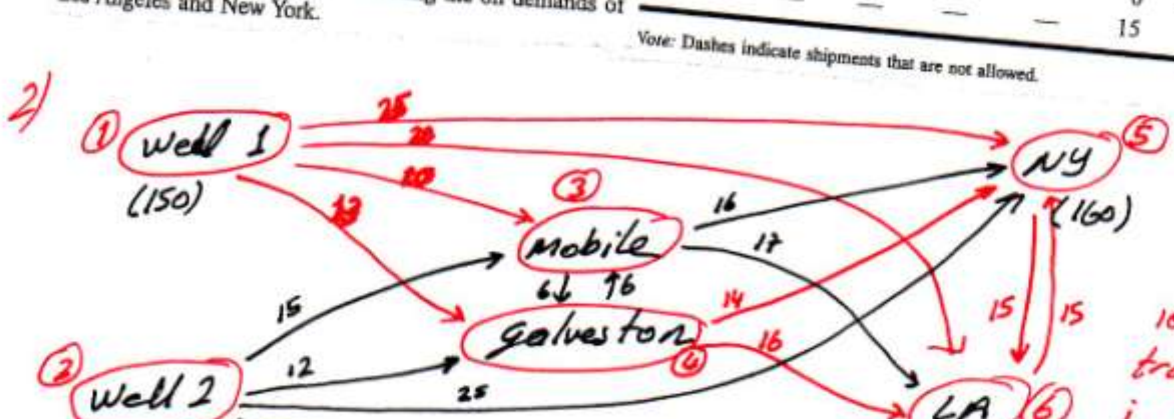
2 Sunco Oil produces oil at two wells. Well 1 can produce as many as 150,000 barrels per day, and well 2 can produce as many as 200,000 barrels per day. It is possible to ship oil directly from the wells to Sunco's customers in Los Angeles and New York. Alternatively, Sunco could transport oil to the ports of Mobile and Galveston and then ship it by tanker to New York or Los Angeles. Los Angeles requires 160,000 barrels per day, and New York requires 140,000 barrels per day. The costs of shipping 1,000 barrels between two points are shown in Table 61. Formulate a transshipment model (and equivalent transportation model) that could be used to minimize the transport costs in meeting the oil demands of Los Angeles and New York.

TABLE 61

From	To ($)					
	Well 1	Well 2	Mobile	Galveston	N.Y.	L.A.
Well 1	0	—	10	13	25	28
Well 2	—	0	15	12	26	25
Mobile	—	—	0	6	16	17
Galveston	—	—	6	0	14	16
N.Y.	—	—	—	—	0	15
L.A.	—	—	—	—	15	0

Note: Dashes indicate shipments that are not allowed.

2)



x_{ij} : # of 1000 barrels trasported from i to j.

Minimize $z = 10x_{13} + 13x_{14} + 25x_{15} + 28x_{16} + 15x_{23} + 12x_{24} + 25x_{25} + 28x_{26}$
$+ 6x_{34} + 16x_{35} + 17x_{36} + 6x_{43} + 14x_{45} + 16x_{46} + 15x_{56} + 15x_{65}$

Subject to;

$x_{13} + x_{14} + x_{15} + x_{16} = 150$

$x_{23} + x_{24} + x_{25} + x_{26} = 200$

$x_{13} + x_{23} + x_{43} = x_{34} + x_{35} + x_{36}$

$x_{14} + x_{24} + x_{34} = x_{43} + x_{45} + x_{46}$

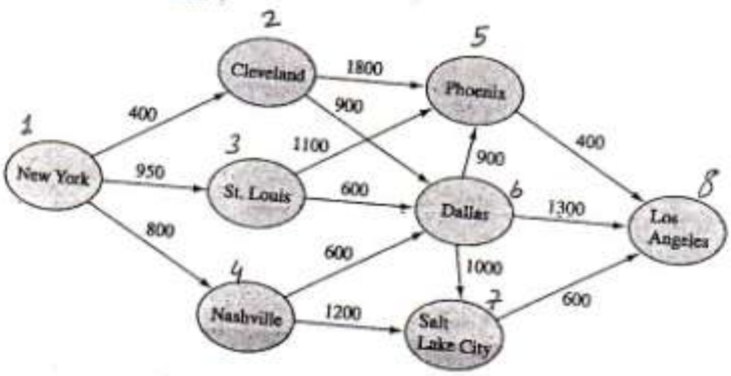
$x_{15} + x_{25} + x_{35} + x_{45} + x_{65} = 160 + x_{56}$

$x_{16} + x_{26} + x_{36} + x_{46} + x_{56} = 140 + x_{65}$

$x_{ij} \geq 0$

* Shortest Path Problem

FIGURE 68
Network for Problem 1



1 A truck must travel from New York to Los Angeles. As shown in Figure 68, a variety of routes are available. The number associated with each arc is the number of gallons of fuel required by the truck to traverse the arc.

a ~~Use Dijkstra's algorithm to~~ find the route from New York to Los Angeles that uses the minimum amount of gas.

b. Find the routes from New York to all other cities that uses each of the the minimum amount of gas.

1) $x_{ij} = \begin{cases} 1 & \text{if the route from } i \text{ to } j \text{ is in the shortest path} \\ 0 & \text{o.w.} \end{cases}$

Minimize $z = 400x_{12} + 950x_{13} + 800x_{14} + 1800x_{25} + 900x_{26}$
$+ 1100x_{35} + 600x_{36} + 600x_{46} + 1200x_{47} + 900x_{56} + 600x_{58}$
$+ 1000x_{67} + 1300x_{68} + 600x_{78}$

a) Subject to;

$x_{12} + x_{13} + x_{14} = 1$

$x_{12} - x_{25} - x_{26} = 0$

$x_{13} - x_{35} - x_{36} = 0$

$x_{14} - x_{46} - x_{47} = 0$

$x_{25} + x_{35} + x_{65} - x_{58} = 0$

$x_{26} + x_{36} + x_{46} - x_{65} - x_{67} - x_{68} = 0$

$x_{47} + x_{67} - x_{78} = 0$

$x_{58} + x_{68} + x_{78} = 1$ $\quad x_{ij} = 0$ or 1

b) Subject to;

$x_{12} + x_{13} + x_{14} = 7$

$x_{12} - x_{25} - x_{26} = 1$

$x_{13} - x_{35} - x_{36} = 1$

$x_{14} - x_{46} - x_{47} = 1$

$x_{25} + x_{35} + x_{65} - x_{58} = 1$

$x_{26} + x_{36} + x_{46} - x_{65} - x_{67} - x_{68} = 1$

$x_{47} + x_{67} - x_{78} = 1$

$x_{58} + x_{68} + x_{78} = 1$

$x_{ij} \geq 0$ and integer.

IP: Integer Programming Problems

EXAMPLE 1 Capital Budgeting IP

Stockco is considering four investments. Investment 1 will yield a net present value (NPV) of \$16,000; investment 2, an NPV of \$22,000; investment 3, an NPV of \$12,000; and investment 4, an NPV of \$8,000. Each investment requires a certain cash outflow at the present time: investment 1, \$5,000; investment 2, \$7,000; investment 3, \$4,000; and investment 4, \$3,000. Currently, \$14,000 is available for investment. Formulate an IP whose solution will tell Stockco how to maximize the NPV obtained from investments 1–4.

EXAMPLE 2 Capital Budgeting (Continued)

Modify the Stockco formulation to account for each of the following requirements:

1 Stockco can invest in at most two investments.

2 If Stockco invests in investment 2, they must also invest in investment 1.

3 If Stockco invests in investment 2, they cannot invest in investment 4.

Example 1

Investment	1	2	3	4
Required Cash	5000	7000	4000	3000
NPV	16000	22000	12000	8000

Available Cash: 14 000

$$x_j = \begin{cases} 1 & \text{if investment } j = 1,2,3,4 \text{ is made} \\ 0 & \text{o.w.} \end{cases}$$

Maximize $z = 16000x_1 + 22000x_2 + 12000x_3 + 8000x_4$

Subject to $5000x_1 + 7000x_2 + 4000x_3 + 3000x_4 \leq 14000$

$x_{ij} = 0$ or 1

Example 2

1) $x_1 + x_2 + x_3 + x_4 \leq 2$

2) $x_1 \geq x_2$

3) $x_2 + x_4 \leq 1$

EXAMPLE 3 Fixed-Charge IP

Gandhi Cloth Company is capable of manufacturing three types of clothing: shirts, shorts, and pants. The manufacture of each type of clothing requires that Gandhi have the appropriate type of machinery available. The machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, \$200 per week; shorts machinery, \$150 per week; pants machinery, \$100 per week. The manufacture of each type of clothing also requires the amounts of cloth and labor shown in Table 2. Each week, 150 hours of labor and 160 sq yd of cloth are available. The variable unit cost and selling price for each type of clothing are shown in Table 3. Formulate an IP whose solution will maximize Gandhi's weekly profits.

TABLE 3
Revenue and Cost Information for Gandhi

	Clothing Type	Sales Price (\$)	Variable Cost (\$)
1.	Shirt	12	6
2.	Shorts	8	4
3.	Pants	15	8

TABLE 2
Resource Requirements for Gandhi

Clothing Type	Labor (Hours)	Cloth (Square Yards)	MACHINERY
Shirt	3	4	200
Shorts	2	3	150
Pants	6	4	100
AVAILABLE	150	160	

Example 3

x_j : Number of clothing type j manufactured

$y_j = \begin{cases} 1 & \text{if any of } x_j \text{ is manufactured} \\ 0 & \text{o.w.} \end{cases}$

Maximize $z = 12x_1 + 8x_2 + 15x_3 - 6x_1 - 4x_2 - 8x_3 - 200y_1 - 150y_2 - 100y_3$

Subject to: $3x_1 + 2x_2 + 6x_3 \leq 150$

$4x_1 + 3x_2 + 4x_3 \leq 160$

$x_1 \leq 40y_1$

$x_2 \leq 53y_2$

$x_3 \leq 25y_3$

$x_j \geq 0$ and integer, $y_j = 0$ or 1

Note that, $40 = $ Max. x_1 that can be produced because

$40 = \text{Min}\left(\left\lceil \frac{150}{3} \right\rceil ; \left\lceil \frac{160}{4} \right\rceil\right)$

Likewise, Max. x_2 is 53 and Max. $x_3 = 25$.

EXAMPLE 4 **The Lockbox Problem**

J. C. Nickles receives credit card payments from four regions of the country (West, Midwest, East, and South). The average daily value of payments mailed by customers from each region is as follows: the West, \$70,000; the Midwest, \$50,000; the East, \$60,000; the South, \$40,000. Nickles must decide where customers should mail their payments. Because Nickles can earn 20% annual interest by investing these revenues, it would like to receive payments as quickly as possible. Nickles is considering setting up operations to process payments (often referred to as lockboxes) in four different cities: Los Angeles, Chicago, New York, and Atlanta. The average number of days (from time payment is sent) until a check clears and Nickles can deposit the money depends on the city to which the payment is mailed, as shown in Table 4. For example, if a check is mailed from the West to Atlanta, it would take an average of 8 days before Nickles could earn interest on the check. The annual cost of running a lockbox in any city is \$50,000. Formulate an IP that Nickles can use to minimize the sum of costs due to lost interest and lockbox operations. Assume that each region must send all its money to a single city and that there is no limit on the amount of money that each lockbox can handle.

TABLE 4
Average Number of Days from Mailing of Payment Until Payment Clears

	To				
From	City 1 (Los Angeles)	City 2 (Chicago)	City 3 (New York)	City 4 (Atlanta)	PAYMENT
Region 1 West	2	6	8	8	70000
Region 2 Midwest	6	2	5	5	50 000
Region 3 East	8	5	2	5	60000
Region 4 South	8	5	5	2	40000

* Earn 20% annual interest

* Cost of running a lockbox : 50 000\$

Example 4

$$x_{ij} = \begin{cases} 1 & \text{if Region } i \text{ payment is sent to City } j \\ 0 & \text{o.w.} \end{cases}$$

$$y_j = \begin{cases} 1 & \text{if lockbox is operated at City } j \\ 0 & \text{o.w.} \end{cases}$$

$$\begin{aligned} \text{Minimize} \quad z = 0{,}20 \cdot \Big[& 70 \cdot (2x_{11} + 6x_{12} + 8x_{13} + 8x_{14}) \\ & + 50 \cdot (6x_{12} + 2x_{22} + 5x_{23} + 5x_{24}) \\ & + 60 \cdot (8x_{31} + 5x_{32} + 2x_{33} + 5x_{34}) \\ & + 40 \cdot (8x_{41} + 5x_{42} + 5x_{43} + 2x_{44}) \Big] \\ + 50 \cdot \big[& y_1 + y_2 + y_3 + y_4 \big] \end{aligned}$$

Subject to;

$$x_{11}+x_{12}+x_{13}+x_{14}=1 \qquad x_{11}+x_{21}+x_{31}+x_{41} \le 4y_1$$
$$x_{21}+x_{22}+x_{23}+x_{24}=1 \qquad x_{12}+x_{22}+x_{32}+x_{42} \le 4y_2$$
$$x_{31}+x_{32}+x_{33}+x_{34}=1 \qquad x_{13}+x_{23}+x_{33}+x_{43} \le 4y_3$$
$$x_{41}+x_{42}+x_{43}+x_{44}=1 \qquad x_{14}+x_{24}+x_{34}+x_{44} \le 4y_4$$

$$x_{ij}=0 \text{ or } 1 \qquad y_j=0 \text{ or } 1$$

EXAMPLE 5 Facility-Location Set-Covering Problem

There are six cities (cities 1–6) in Kilroy County. The county must determine where to build fire stations. The county wants to build the minimum number of fire stations needed to ensure that at least one fire station is within 15 minutes (driving time) of each city. The times (in minutes) required to drive between the cities in Kilroy County are shown in Table 6. Formulate an IP that will tell Kilroy how many fire stations should be built and where they should be located.

TABLE 6
Time Required to Travel between Cities in Kilroy County

From	To: City 1	City 2	City 3	City 4	City 5	City 6
City 1	0	10	20	30	30	20
City 2	10	0	25	35	20	10
City 3	20	25	0	15	30	20
City 4	30	35	15	0	15	25
City 5	30	20	30	15	0	14
City 6	20	10	20	25	14	0

Example 5

$$x_j = \begin{cases} 1 & \text{if fire station is built to city } j=1,2,\ldots,6 \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Minimize } z = x_1+x_2+x_3+x_4+x_5+x_6$$

Subject to
$$x_1+x_2 \ge 1 \rightarrow \text{City 1}$$
$$x_1+x_2+x_6 \ge 1 \rightarrow \text{City 2}$$
$$x_3+x_4 \ge 1 \rightarrow \text{City 3}$$
$$x_3+x_4+x_5 \ge 1 \rightarrow \text{City 4}$$
$$x_4+x_5+x_6 \ge 1 \rightarrow \text{City 5}$$
$$x_2+x_5+x_6 \ge 1 \rightarrow \text{City 6}$$

Fire stations that have driving time ≤15

$$x_j = 0 \text{ or } 1$$

EXAMPLE 6 **Either-Or Constraint**

Dorian Auto is considering manufacturing three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by, each type of car are shown in Table 8. Currently, 6,000 tons of steel and 60,000 hours of labor are available. For production of a type of car to be economically feasible, at least 1,000 cars of that type must be produced. Formulate an IP to maximize Dorian's profit.

TABLE 8
Resources and Profits for Three Types of Cars

	Car Type			
Resource	(x_1) Compact	(x_2) Midsize	(x_3) Large	AVAILABLE
Steel required	1.5 tons	3 tons	5 tons	6000 tons
Labor required	30 hours	25 hours	40 hours	60 000 hours
Profit yielded ($)	2,000	3,000	4,000	

*If a car is produced, produce at least 1000 cars.

If either $f(x_1, x_2, \ldots, x_n) \le 0$ or $g(x_1, x_2, \ldots, x_n) \le 0$ must be satisfied, we use the constraints:

$$f(x_1, x_2, \ldots, x_n) \le My$$
$$g(x_1, x_2, \ldots, x_n) \le M(1-y)$$

where $y = 0$ or 1 and M is a sufficiently large number.

Example 6 x_j: # of type j cars produced.

We have, $\underbrace{x_1 \le 0}_{f}$ or $x_1 \ge 1000 \Rightarrow \underbrace{1000 - x_1 \le 0}_{g}$. Then;

$$x_1 \le 2000 y_1$$
$$1000 - x_1 \le 2000(1-y_1)$$

Since we can produce maximum $M = 2000$ Compact type.

Likewise, we'll use $x_2 \le 2000 y_2$ and $x_3 \le 1200 y_3$

$$1000 - x_2 \le 2000(1-y_2) \qquad 1000 - x_3 = 1200(1-y_3)$$

Maximize $z = 2000x_1 + 3000x_2 + 4000x_3$

Subject to:

$$1{,}5x_1 + 3x_2 + 5x_3 \le 6000 \qquad x_2 \le 2000 y_2$$
$$30x_1 + 25x_2 + 40x_3 \le 60000 \qquad 1000 - x_2 \le 2000(1-y_2)$$
$$x_1 \le 2000 y_1 \qquad x_3 \le 1200 y_3$$
$$1000 - x_1 \le 2000(1-y_1) \qquad 1000 - x_3 \le 1200(1-y_3)$$
$$x_j \ge 0 \text{ and integer} \quad y_j = 0 \text{ or } 1$$

EXAMPLE 7 **IP with Piecewise Linear Functions**

Euing Gas produces two types of gasoline (gas 1 and gas 2) from two types of oil (oil 1 and oil 2). Each gallon of gas 1 must contain at least 50 percent oil 1, and each gallon of gas 2 must contain at least 60 percent oil 1. Each gallon of gas 1 can be sold for 12¢, and each gallon of gas 2 can be sold for 14¢. Currently, 500 gallons of oil 1 and 1,000 gallons of oil 2 are available. As many as 1,500 more gallons of oil 1 can be purchased at the following prices: first 500 gallons, 25¢ per gallon; next 500 gallons, 20¢ per gallon; next 500 gallons, 15¢ per gallon. Formulate an IP that will maximize Euing's profits (revenues − purchasing costs).

Example 7

	Gas 1	Gas 2	AVAILABLE
Oil 1	x_{11}	x_{12}	500
Oil 2	x_{21}	x_{22}	1000
	$\geq 0{,}5$ Oil 1	$\geq 0{,}6$ Oil 1	
Price	12¢	14¢	

x: Oil 1 Purchased.

We can purchase Oil 1 with the cost function $c(x)$

$$c(x) = \begin{cases} 25x & \overset{=b_1}{0} \leq x \leq \overset{=b_2}{500} \\ 2500 + 20x \;\; (= 25.500 + 20.(x-500)) & \overset{=b_2}{500} \leq x \leq \overset{=b_3}{1000} \\ 7500 + 15x \;\; (= 2500 + 20.1000 + 15.(x-1000)) & \overset{=b_3}{1000} \leq x \leq \overset{=b_4}{1500} \end{cases}$$

Maximize $z = 12(x_{11} + x_{21}) + 14(x_{12} + x_{22}) - c(x)$

subject to

$$\frac{x_{11}}{x_{11} + x_{21}} \geq 0{,}5 \qquad x_{11} + x_{12} \leq 500 + x$$

$$x_{21} + x_{22} \leq 1000$$

$$\frac{x_{12}}{x_{12} + x_{22}} \geq 0{,}6 \qquad x \leq 1500$$

$$x_{ij} \geq 0 \quad x \geq 0$$

* But this is NOT an LP model (so, it can NOT be an IP) because $c(x)$ is NOT a linear function (it is piecewise linear)

We make a piecewise linear function a linear function using binary variables by the following steps:

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Step 1: Replace $f(x)$ by $z_1 f(b_1) + z_2 f(b_2) + \dots + z_n f(b_n)$ where $f(x)$ is a piecewise linear function such that

$$f(x) = \begin{cases} f_1(x) & b_1 \le x \le b_2 \\ f_2(x) & b_2 \le x \le b_3 \\ \vdots & \vdots \\ f_{n-1}(x) & b_{n-1} \le x \le b_n \end{cases}$$

Step 2: Add following constraints to the problem:

$z_1 \le y_1$; $z_2 \le y_1 + y_2$; $z_3 \le y_2 + y_3$; ... ; $z_{n-1} \le y_{n-2} + y_{n-1}$; $z_n \le y_{n-1}$

$y_1 + y_2 + \dots + y_{n-1} = 1$ $\quad$ $y_i = 0$ or 1 $\quad i = 1, 2, \dots, n-1$

$z_1 + z_2 + \dots + z_n = 1$ $\quad$ $z_i \ge 0$ $\quad i = 1, 2, \dots, n$

$x = z_1 b_1 + z_2 b_2 + \dots + z_n b_n$

* So, we make our cost function linear as follows;

Step 1: $c(x) = z_1 \underbrace{c(0)}_{=0} + z_2 \underbrace{c(500)}_{=12500} + z_3 \underbrace{c(1000)}_{=22500} + z_4 \underbrace{c(1500)}_{=30000}$

$c(x) = 0 \cdot z_1 + 12500 \cdot z_2 + 22500 \cdot z_3 + 30000 \cdot z_4$

Step 2: $z_1 \le y_1 \quad z_2 \le y_1 + y_2 \quad z_3 \le y_2 + y_3 \quad z_4 \le y_3$

$z_1 + z_2 + z_3 + z_4 = 1$ $\quad$ $y_i = 0$ or 1 $\quad i = 1, 2, 3$

$y_1 + y_2 + y_3 = 1$ $\quad$ $z_i \ge 0$ $\quad i = 1, 2, 3, 4$

$x = 0 z_1 + 500 z_2 + 1000 z_3 + 1500 z_4$

* The idea works as follows:

Let $z_2 = 0{,}2$ and $z_3 = 0{,}8$ (so, $y_2 = 1$ and $y_1 = y_3 = 0$)

Then, $x = 0{,}2 \cdot 500 + 0{,}8 \cdot 1000 = 900$ (z_2 and z_3 are unique for $x = 900$ $(z_3 = 1 - z_2)$)

And $c(x) = c(900) = 2500 + 20 \cdot 900 = 20500$

But also $c(900) = z_2\, c(500) + z_3\, c(1000) = 0{,}2 \cdot 12500 + 0{,}8 \cdot 22500 = 20500$

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* Finally, our IP is;

Maximize $z = 12(x_{11}+x_{21}) + 14(x_{12}+x_{22}) - (0z_1 + 12500z_2 + 22500z_3 + 30000z_4)$

Subject to:

$\dfrac{x_{11}}{x_{11}+x_{21}} \geq 0,5$ $\quad z_1 \leq y_1 \quad z_2 \leq y_1+y_2 \quad z_3 \leq y_2+y_3 \quad z_4 \leq y_3$

$z_1+z_2+z_3+z_4 = 1$

$\dfrac{x_{21}}{x_{12}+x_{22}} \geq 0,6$ $\quad y_1+y_2+y_3 = 1$

$x = 0z_1 + 500z_2 + 1000z_3 + 1500z_4$

$x_{11}+x_{12} \leq 500+x$ $\quad y_i = 0$ or 1 $\; i=1,2,3$; $z_i \geq 0$ $\; i=1,2,3,4$

$x_{21}+x_{22} \leq 1000$ $\quad x_{ij} \geq 0$ (note that $0 \leq x \leq 1500$ is already guaranteed)

EXAMPLE 2 Odds and Evens

Woodco sells 3-ft, 5-ft, and 9-ft pieces of lumber. Woodco's customers demand 25 3-ft boards, 20 5-ft boards, and 15 9-ft boards. Woodco, who must meet its demands by cutting up 17-ft boards, wants to minimize the waste incurred. Formulate an LP to help Woodco accomplish its goal, and solve the LP by column generation.

Example 2 →

Type	Demand
3ft	25
5ft	20
9ft	15

* Meet demands by cutting up 17ft boards

x_j: combination used to cut 17ft boards $j = 1,2,\dots,6$

All Possible Combinations	3ft	5ft	9ft	Waste
(x_1) 1	5	0	0	2
(x_2) 2	4	1	0	0
(x_3) 3	2	2	0	1
(x_4) 4	2	0	1	2
(x_5) 5	1	1	1	0
(x_6) 6	0	3	0	2

Minimize

$z = 2x_1 + x_3 + 2x_4 + x_6$

Subject to

$5x_1 + 4x_2 + 2x_3 + 2x_4 + x_5 \geq 25$

$x_2 + 2x_3 + x_5 + 3x_6 \geq 20$

$x_4 + x_5 \geq 15$

$x_j \geq 0$ and integer $j = 1,2,\dots,6$

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